Theory of localization and resonance phenomena in the quantum kicked rotor

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We present an analytic theory of quantum interference and Anderson localization in the quantum kicked rotor (QKR). The behavior of the system is known to depend sensitively on the value of its effective Planck's constant $\tilde{h}$. We here show that for rational values of $\tilde{h}/(4\pi) = p/q$, it bears similarity to a disordered metallic ring of circumference q and threaded by an Aharonov-Bohm flux. Building on that correspondence, we obtain quantitative results for the time–dependent behavior of the QKR kinetic energy, $E(\tilde{t})$ (this is an observable which sensitively probes the system's localization properties). For values of q smaller than the localization length ξ, we obtain scaling $E(\tilde{t}) \sim \Delta\tilde{t}^2$, where $\Delta = 2\pi/q$ is the quasi–energy level spacing on the ring. This scaling is indicative of a long time dynamics that is neither localized nor diffusive. For larger values $q \gg \xi$, the functions $E(\tilde{t}) \to \xi^2$ saturates (up to exponentially small corrections $\sim \exp(-q/\xi)$), thus reflecting essentially localized behavior.

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I. INTRODUCTION

The kicked rotor, or standard map [1] is one of the most prominent model systems of nonlinear dynamics. In spite of its very simple construction – the rotor Hamiltonian depends on just a single parameter, the dimensionless kicking strength, K – it shows complex physical behavior. Specifically, the classical rotor undergoes a transition from integrable to chaotic dynamics as the kicking strength is increased. Quantization in the chaotic regime leads to a quantum system that bears strong similarity to a disordered multi-channel quantum wire. Much like in these systems, the dynamics of the quantum kicked rotor (QKR) is governed by mechanisms of quantum interference and localization [2, 3]. The applied interest in researching these structures has increased since the mid nineties [4] when concrete realizations of the rotor in atom optics settings became an option [5].

The analogies to condensed matter systems notwithstanding, there are also a number of important differences. The QKR is not a genuinely disordered system. Rather, its effective stochasticity roots in mechanisms of incommensurability. The differences from generic disorder manifest themselves in anomalies showing as Planck's constant assumes peculiar "resonant" values [2, 6–9]. (In this paper, we will not discuss the role of *classical* anomalies, such as accelerator mode formation [10] at special values of K.) At these values, the system becomes effectively periodic (in the space of angular momentum quantum numbers) and generally escapes localization. Various aspects of QKR resonances have been discussed before in terms of general observations, phenomenological reasoning and direct diagonalization (see Ref. [8] for review.) However, a microscopic theory of the interplay of resonances and localization has not yet been formulated. The construction of such a theory is the subject of the present paper.

Below, we will introduce an analytic and parametrically controlled theory of localization phenomena in the QKR both on and off resonant values of Planck's constant, $\tilde{h}$. It turns out that at resonant values, $\tilde{h} = 4\pi p/q$, with p, q coprime numbers, localization phenomena in the QKR can be described in terms of a field theory defined on a ring subject to an Aharonov-Bohm flux (the latter implementing the Bloch phases pervasive in the physics of periodically extended systems.) We will apply perturbative and non–perturbative methods to analyze this field theory in terms of different observables. Emphasis will be put on the discussion of the time dependent increase in the rotor's kinetic energy, an observable that carries immediate information on the transport characteristics of the system.

Before turning to a qualitative discussion of results, it is worthwhile to put the present theory into the context of earlier work. The connections to disordered systems were first discussed in Ref. [11]. That work introduced a mapping of the QKR Hamiltonian onto an effectively disordered tight binding system. However, the long-ranged correlated 'disorder' inherent to that model made it difficult to treat analytically. A different approach was introduced in Ref. [12], where a diagrammatic approach conceptually similar to the impurity diagram language of disordered systems was introduced. In this work, perturbative quantum interference processes ("weak localization") were discussed in terms of low energy fluctuations in angular momentum space (for a refined implementation of the diagrammatic approach, see Ref. [13].) These results inspired a subsequent field theoretical study [14] in which the large scale dynamics of the quantum kicked rotor was mapped onto a one-dimensional supersymmetric nonlinear σ–model (otherwise known as the effective field theory of Anderson localization in disordered quasi-one-dimensional wires [15].)

None of these works payed attention to the resonance phenomena at rational $\tilde{h}/(4\pi)$, a lack of resolution that has prompted some criticism [9, 16]. Specifically, it has been argued that the insensitivity of earlier work to the algebraic characteristics of $\tilde{h}$ might be indicative of ad hoc assumptions hidden in the construction. One of the motivations

behind the present work is to show that these objections have been ill founded. Rather, we will see that the field theory below – by and large a refined variant of the earlier theory [14] – is capable of describing the system from large scales down to the fine structure of individual quasi-energy levels.

Nonetheless, it is worthwhile to point out that our theory is far from 'mathematically rigorous'. Rather, it is based on a mapping of QKR correlation functions onto a functional integral (exact) followed by an identification of field configurations of least action, plus an integration over fluctuations. The last two steps are approximate, if parametrically controlled (by a small parameter $\tilde{h}/K$.) In this sense, our theory below falls into the general framework of semiclassical approaches to nonlinear systems. The semiclassical limit enables us to explicitly describe system observables which more rigorous lines of reasoning [17] can only characterize in implicit terms. We also emphasize that no averaging procedures other than a global average over the quasi-energy spectrum will be performed throughout.

The rest of this paper is organized as follows: we start with a qualitative discussion of the system and a summary of our main results (Section II). In Section III we introduce the functional integral approach to the problem, which will be reduced to an effective semiclassical action in Section IV. The low energy physics of this action will then be discussed in Sections V (off-resonance) and VI (on resonance). We summarize in Section VII. Some technical details are deferred to Appendices A-C.

II. QUALITATIVE DISCUSSIONS

The QKR is a periodically driven ("kicked") quantum particle moving on a circle. In units where the particle mass and the circular radius are set to unity, its Hamiltonian reads

$$\hat{H}(t) \;=\; \frac{\hat{l}^2}{2} + k\cos\hat{\theta}\,\sum_m \delta(t-mT), \tag{1}$$

where T and k are the period and the kicking strength, respectively. The angular operator, $\hat{\theta}$, and the angular momentum operator, $\hat{l}$ obey the canonical quantization condition $[\hat{\theta}\,,\hat{l}] = i\hbar$ where $\hat{l} = i\hbar\frac{\partial}{\partial\theta}$. In rescaled dimensionless time, $t \to t/T \equiv \tilde{t}$, the dynamics of the system is determined by two parameters: the effective Planck's constant $\tilde{h} = \hbar T$ and the classical nonlinear parameter $K = kT$. (In the chosen units, both parameters are dimensionless.)

A. Symmetries

At integer times $\tilde{t} = 0, 1, 2, \ldots$, the solution of the Schrödinger equation $i\tilde{h}\,\partial_{\tilde{t}}|\psi(\tilde{t})\rangle = T^2\hat{H}|\psi(\tilde{t})\rangle$ can be expressed as $|\psi(\tilde{t})\rangle = \hat{U}^{\tilde{t}}|\psi(0)\rangle$, where $\hat{U}$ is the Floquet operator, i.e. the unitary operator governing the time evolution during one elementary time step:

$$\hat{U} = \exp\left(\frac{i\tilde{h}\hat{n}^2}{4}\right)\exp\left(\frac{iK\cos\hat{\theta}}{\tilde{h}}\right)\exp\left(\frac{i\tilde{h}\hat{n}^2}{4}\right), \tag{2}$$

where we have introduced $\hat{n} \equiv \hat{l}/\hbar$ as a rescaled angular momentum operator with integer–valued spectrum quantum.

For Planck's constants commensurable with 4π, $\tilde{h} = 4\pi p/q$, with coprime p, $q \in \mathbb{N}$, the Floquet operator is invariant under the shift $\hat{n} \to \hat{n} + q$, or

$$[\hat{U}, \hat{T}_q] = 0, \tag{3}$$

where $\hat{T}_q$ is the translation operator by q steps, $\hat{T}_q|n\rangle = |n+q\rangle$ on eigenstates $\hat{n}|n\rangle = n|n\rangle$. This discrete translational symmetry implies that the (quasi)energy states of the Floquet operator can be expanded in a basis of Bloch states,

$$e^{i\theta n}\psi_{n,\theta},$$

where $\theta \in [0, 2\pi/q]$ and $\psi_{n+q,\theta} = \psi_{n,\theta}$. The infinite extension of Bloch states in angular momentum space means the absence of localization at these "resonant values" of $\tilde{h}$. [2, 6–9] (However, for periodicity intervals q much larger than the intrinsic localization length of the system, the wave function amplitudes at the boundaries of the "unit cell" are exponentially small, which means that the system behaves effectively localized.) Below, we will explore the interplay of localization and Bloch periodicity in some detail. Our discussion will include the case of *irrational* $\tilde{h}/4\pi$ as a limit

of co–prime values p/q with diverging q. We will assume that $q \gg K/\tilde{h} \equiv \ell$, which means that the angular momentum unit cell is larger than the length scales at which diffusive dynamics begins to form.

In dirty metals, the manifestations of localization depend sensitively on the presence or absence of time reversal, $t \to -t, r \to r, p \to -p$ (where r and p are coordinate and momentum respectively.) In the context of the kicked rotor, momentum and coordinate space change their roles, which is why time reversal followed by space inversion, $\mathrm{T}_c : t \to -t, \theta \to -\theta, l \to l$ becomes a relevant symmetry. It has been shown [12, 13, 18] that T_c plays a role analogous to time reversal in metals. The Hamiltonian (1) is T_c–invariant, and in our analysis below, we need to take this symmetry into account. (T_c–invariance may be broken e.g. by a shift $\cos(\hat{\theta}) \to \cos(\hat{\theta}+a)$, $\quad a = \mathrm{const.}$ However, in this paper, we focus on the invariant system.) In a basis of angular momentum eigenstates, $\{|n\rangle\}$, $\hat{l}|n\rangle = \tilde{h}n\,|n\rangle$, $n \in \mathbb{Z}$, the symmetry T_c acts by matrix transposition $A_{nn'} \to (A^T)_{nn'} \equiv A_{n'n}$ (much like conventional time reversal acts by transposition in a real space basis.)

B. Off resonance

Let us temporarily assume that $\tilde{h} \in 4\pi\mathbb{R}\backslash\mathbb{Q}$ is irrational (or rational with $q \gg \xi$, so that the periodicity of the system plays no role.) Throughout this paper, we consider the system prepared in a pure initial state with zero angular momentum, i.e., $|0\rangle\langle 0|$. Under the Floquet dynamics it will evolve into $(\hat{U})^{\tilde{t}}|0\rangle\langle 0|(\hat{U}^\dagger)^{\tilde{t}}$. Its (squared) deviation from the point of departure is measured by the expectation value $E(\tilde{t}) \equiv \frac{1}{2}\mathrm{tr}(\hat{n}^2(\hat{U})^{\tilde{t}}|0\rangle\langle 0|(\hat{U}^\dagger)^{\tilde{t}})$. Comparison with the Hamiltonian (1) shows that this quantity may be interpreted in terms of the system's change in kinetic energy. In the classical limit, $E(\tilde{t})$ value grows indefinitely in time, a phenomenon that may be interpreted as a mechanism of 'heating'. More precisely,

$$E(\tilde{t}) \sim D_0\tilde{t},$$

where $D_0 \equiv (K/2\tilde{h})^2$ has the status of a diffusion constant in angular momentum space. Notice that the energy expectation value can be represented as $E(\tilde{t}) = \frac{1}{2}\sum_n n^2 K(n,0;\tilde{t})$, where $K(n,n';t)$ is the density-density correlation function in angular momentum space (for a precise definition, see Section III below.) The linear growth in time is indicative of diffusive behavior $K(q,\omega) \sim (-i\omega + D_0q^2)^{-1}$, where (ω, q) is Fourier conjugate to $(\tilde{t}, n)$. Empirically, it has long been known [3] that in the *quantum* model, the linear growth in energy comes to an end at times $\tilde{t}_L \sim D_0$. Simple scaling arguments show that this saturation implies a crossover $D_0 \to D(\omega) \sim i\omega$ at frequencies $\omega \lesssim D_0^{-1}$. The scaling $D(\omega) \sim i\omega$ is a manifestation of quantum localization [15].

A microscopic quantum theory of the system must be capable of establishing diffusive dynamics at intermediate time scales, and to describe the quantum interference processes rendering the diffusion constant frequency dependent.

C. On resonance

We now discuss what happens for rational values $\tilde{h} = 4\pi p/q$. The commutativity (3) means that we are considering a variant of a q-periodic quantum system. For the convenience of the reader, a few basic structures of periodically extended quantum systems are recapitulated in Fig. 1. The Bloch function basis can be understood as a basis whose elements exhibit a definite phase change $\theta q \in [0, 2\pi]$ across a single unit cell (top panel.) For a given θ, one may then consider the system within the reduced scheme of just a single unit cell, subject to twisted boundary conditions $\psi(0) = \psi(q)\exp(i\theta q)$ (middle left). Equivalently, one may interpret the system in terms of an Aharonov-Bohm ring (middle right) subject to a gauge flux θq. The q energy levels $\epsilon_j(\theta)$ of this system (bottom) form a $2\pi/q$ periodic family, each j defines a Bloch band.

Pioneering work on the physics of the kicked rotor at these "resonant" values of Planck's constant has been done by Izrailev and Shepelyansky (IS) [6] (see Ref. [8] and Section VI A below for review.) In this work, it has been argued that at resonance, the energy stored in the system increases as

$$E(\tilde{t}) = \eta\tilde{t}^2 + a\tilde{t} + b(\tilde{t}), \tag{4}$$

(designation of constants η, a and function $b(\tilde{t})$ taken from the review Ref. [8]). Let us briefly discuss the meaning of this expansion, as discussed in IS. Eq. (4) has the status of a large time asymptotic. For times large enough such that $\tilde{t}^{-1}$ is smaller than the level spacing of the unit cell, $\Delta = 2\pi/q$, we are in a "deep quantum regime", where transitions between individual Bloch bands can be neglected. In this regime, the energy increase is governed by expressions such as (symbolic notation)

$$E(\tilde{t}) \sim \int d\theta d\theta'\, e^{i(\epsilon_j(\theta)-\epsilon_j(\theta'))\tilde{t}-in(\theta-\theta')} n^2 \sim \tilde{t}^2.$$

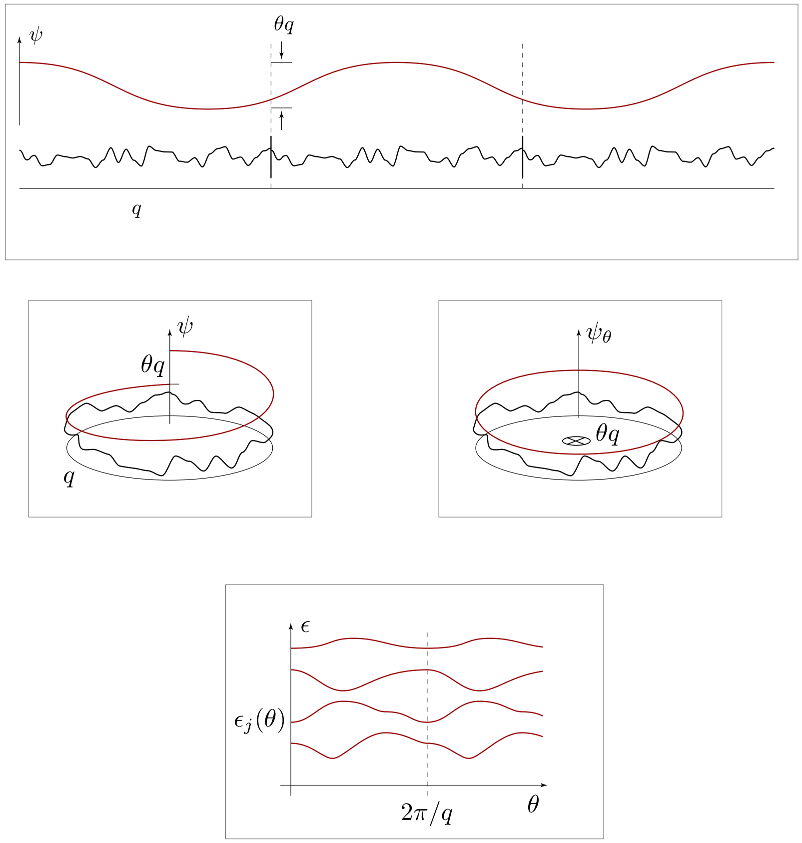

FIG. 1: Wave functions and spectrum of the periodically extended system.

Thus, the $\tilde{t}^2$ contribution is a measure of the Bloch–phase auto–correlation of individual levels. The auto–correlation of levels also yields a quantum correction linear in $\tilde{t}$, and a residual contribution $b(\tilde{t})$ of indefinite time dependence. The coefficients of both terms, η and a, reflect the dispersion of Bloch bands, and vanish if bands are flat. Finally, in regimes with localization, $\xi \ll q$ we expect the increase in n^2 to come to an end at large times $\tilde{t} \sim \xi^2/D_0$. At larger times, $E(\tilde{t})$ will saturate. (With exponentially small corrections $\sim \exp(-q/\xi)$ proportional to the wave function overlap across the ring.)

The construction of IS that led to (4) has been formal in that it relied on explicit knowledge of eigenfunctions and -values of the Floquet operator; apart from a few exceptional values of q, no concrete results for the coefficients of the expansion have been available. Equally important, at times shorter than Δ^{-1} mechanisms of energy diffusion different from those discussed in IS play a role. Specifically, for small values of time, the finite extension of the unit cell is not yet felt, and angular momentum will diffuse as in an infinite system. This generates a (leading) $\sim t$ dependence in $b(\tilde{t})$, which is purely *classical*. (Within the framework of a quantum approach, diffusion has to do with inter–band transitions.) At short times, the profile of $E(\tilde{t})$ will, thus, be different from (4).

Below, we will analytically derive the function $E(\tilde{t})$ for periodicity volumes obeying $\ell < q < \xi$. As a result, we obtain a universal scaling function

$$
\begin{aligned}
E(\tilde{t}) &= D_q q\, F\left(\frac{\tilde{t}}{q}\right), \\
F(x) &= \begin{cases} x + \frac{x^3}{3}, & (qE_c)^{-1} \ll x < 1 \\ x^2 + \frac{1}{3}, & x > 1 \end{cases},
\end{aligned}
\tag{5}
$$

where $E_c = D_q/q^2$ is the inverse of the classical diffusion time through the periodicity volume. Referring for a more substantial discussion to Section VI B, we here merely note that Eq. (5) predicts diffusive scaling $E \sim D_q \tilde{t}$ for times $\tilde{t} < \Delta^{-1}$. For larger times, we obtain a quadratic increase, $E \sim \eta \tilde{t}^2$, governed by the universal coefficient $\eta = D_q/q$. For the discussion of the meaning of this coefficient and of the correction terms in (5), we refer to section VI B.

In Section VI C, we will briefly discuss the regime of large resonance volumes $q > \xi$. Our results are in line with the general expectations formulated above.

D. Close to fundamental resonance

The sensitivity of the rotor to the algebraic properties of $\tilde{h}$ manifests itself not only on resonances, but also in the close vicinity of low–order resonant values. To be specific, consider values $\tilde{h} = 4\pi p + \epsilon$, close to the fundamental resonance $q = 1$. Straightforward substitution into (2) shows that the Floquet operator defined for the pair $(\tilde{h}, K)$, is identical to one with $(\tilde{h}_\epsilon, K_\epsilon)$, where [19]

$$\tilde{h}_\epsilon \equiv \epsilon, \qquad K_\epsilon \equiv \frac{K\epsilon}{4\pi p + \epsilon}. \tag{6}$$

This duality phenomenon has been dubbed "ϵ-classics": a rotor with large value of Planck's constant becomes equivalent to one with small $\tilde{h} \sim \epsilon$. Faithfulness to this symmetry is a benchmark criterion which the present approach obeys. In practice, this means that system characteristics such as the localization length will depend on $\tilde{h}$ and K through certain invariant combinations (that have been conjectured phenomenologically before [20].)

We now turn to the quantitative formulation of our theory.

III. FUNCTIONAL INTEGRAL FORMULATION

As in condensed matter physics, localization properties of the QKR can be conveniently probed in terms of "two particle Green functions". In this paper, emphasis will be put on the density correlation function

$$K_\omega(n_1, n_2) \equiv \sum_{\tilde{t},\tilde{t}'=0}^{+\infty} \left\langle \langle n_1|(\hat{U}e^{i\omega_+})^{\tilde{t}}|n_2\rangle\langle n_2|(\hat{U}e^{i\omega_-})^{-\tilde{t}'}|n_1\rangle \right\rangle_{\omega_0}, \tag{7}$$

where the frequency arguments, $\omega_\pm \equiv \omega_0 \pm \frac{\omega}{2}$, with ω understood as $\omega + i\delta, \delta \searrow 0$, and we have introduced an average over the quasienergy spectrum, $\langle \cdot \rangle_{\omega_0} := \int_0^{2\pi} \frac{d\omega_0}{2\pi}(\cdot)$. Summation over $\tilde{t}, \tilde{t}'$ obtains

$$K_\omega(n_1, n_2) = \left\langle \langle n_1|\hat{G}^+(\omega_+)|n_2\rangle\langle n_2|\hat{G}^-(\omega_-)|n_1\rangle \right\rangle_{\omega_0}, \tag{8}$$

where

$$\hat{G}^\pm(\omega_\pm) = \sum_{\tilde{t}=0}^{\infty} \left(e^{i\omega_\pm}\hat{U}\right)^{\pm\tilde{t}} = \frac{1}{1 - (e^{i\omega_\pm}\hat{U})^{\pm 1}}, \tag{9}$$

and the $+$ $(-)$ superscript designates the retarded (advanced) Green function. Physically, the function $K_\omega(n_1, n_2)$ describes the probability of propagation $n_2 \to n_1$ in time $\sim \omega^{-1}$. Exponential decay of $(-i\omega K_\omega)$ at $\omega \to 0$ signals the onset of localization. The density correlation function also contains information on the time dependent expectation value of the energy. Comparing with the definition given in the beginning of Section II B, it is straightforward to show

$$E(\tilde{t}) = \frac{1}{2}\sum_n n^2 \int \frac{d\omega}{2\pi} e^{-i\omega\tilde{t}} K_\omega(n, 0). \tag{10}$$

In the present formalism, the consequences of time reversal symmetry are best exposed by "taking the square root" of the Floquet operator,

$$\hat{U}^{\pm 1} = \hat{V}^\pm \hat{V}^{\pm T},$$

where

$$\hat{V}^\pm \equiv \exp\left(\pm\frac{i\tilde{h}\hat{n}^2}{4}\right)\exp\left(\pm\frac{iK\cos\hat{\theta}}{2\tilde{h}}\right). \tag{11}$$

We next introduce the matrix Green function

$$\tilde{G}^\pm \equiv \begin{pmatrix} 1 & e^{\pm\frac{i\omega_\pm}{2}}\hat{V}^\pm \\ e^{\pm\frac{i\omega_\pm}{2}}\hat{V}^{\pm T} & 1 \end{pmatrix}^{-1}. \tag{12}$$

We will refer to the newly introduced two-component structure as "time reversal space", or "T"–space and label it by indices $t, t', \ldots$. The matrix $\tilde{G}$ is related to the original Green function as $\hat{G} = \tilde{G}_{11}$, i.e.

$$K_\omega(n_1, n_2) = \left\langle \langle n_1|\tilde{G}^+_{11}|n_2\rangle\langle n_2|\tilde{G}^-_{11}|n_1\rangle \right\rangle_{\omega_0}. \tag{13}$$

As a first step towards the construction of a field integral formulation, we introduce a superfield

$$\psi \equiv \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \qquad \bar{\psi} \equiv (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow),$$

where component structure refers to the matrix structure in (12), and the superfields $\psi_{\uparrow\downarrow} = \{\psi_{\uparrow\downarrow,\alpha\lambda n}\}$ carry complex commuting (anticommuting) components $\psi_{\alpha=1}$ ($\psi_{\alpha=2}$). The index λ will be used to discriminate between the retarded ($\lambda = +$) and the advanced ($\lambda = -$) sector of the theory. Finally, the commuting variables $\bar{\psi}_{\alpha=1} = \psi^*_{\alpha=1}$ are the complex conjugate of $\psi_{\alpha=1}$ while in the anticommuting sector $\bar{\psi}_{\alpha=2}$ and $\psi_{\alpha=2}$ are independent variables.

We next define the compound Green function

$$G = E^{11}_{\rm AR} \otimes \mathbb{1}_{\rm BF} \otimes \tilde{G}^+(\omega_+) + E^{22}_{\rm AR} \otimes \mathbb{1}_{\rm BF} \otimes \tilde{G}^-(\omega_-)\,, \tag{14}$$

where matrices with subscript "AR" ("BF") act in the two-dimensional spaces of λ (α) indices. Matrices $E^{ij}_X, X = \mathrm{AR}, \mathrm{BF}$ contain zeroes everywhere except for a unity at position (i, j) and $\mathbb{1}_X$ are unit matrices. The correlation function K_ω can then be represented as a Gaussian integral

$$K_\omega(n_1, n_2) = \left\langle \int D(\bar{\psi}, \psi) \exp\left(-\bar{\psi} G^{-1} \psi\right) \mathcal{F}[\bar{\psi}_\uparrow, \psi_\uparrow] \right\rangle_{\omega_0}, \tag{15}$$

where the pre–exponential factor

$$\mathcal{F}[\bar{\psi}, \psi] \equiv \bar{\psi}_{\uparrow 1+n_2} \psi_{\uparrow 1+n_1} \bar{\psi}_{\uparrow 1-n_1} \psi_{\uparrow 1-n_2}.$$

The absence of a normalization factor in (15) follows from the fact that the integration over anticommuting variables normalizes the integral to unity,

$$\mathcal{Z} \equiv \left\langle \int D(\bar{\psi}, \psi) \exp\left(-\bar{\psi} G^{-1} \psi\right) \right\rangle_{\omega_0} = 1.$$

We next reformulate the field integral in a way that will later enable us to make the consequences of the symmetries of the problem manifest. Beginning with time reversal, T_c, we use the symmetry of the Green function, $\tilde{G} = \tilde{G}^T$ to write

$$\bar{\psi} G^{-1} \psi = \frac{1}{2}\left(\bar{\psi} G^{-1} \psi + \psi^T \sigma^3_{\rm BF} G^{-1} \bar{\psi}^T\right) \equiv \bar{\Psi} G^{-1} \Psi,$$

where we defined

$$\bar{\Psi} \equiv \frac{1}{\sqrt{2}}(\bar{\psi}, \psi^T \sigma^3_{\rm BF}), \quad \Psi \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}. \tag{16}$$

and the Pauli matrix $\sigma^3_{\rm BF}$ accounts for the anti–commutativity of the variables $\psi_{\alpha=2}$, i.e. $\psi_{\uparrow,2\lambda n}\psi_{\uparrow,2\lambda' n'} = -\psi_{\uparrow,2\lambda' n'}\psi_{\uparrow,2\lambda n}$, and the same for $\psi_\downarrow$. The newly defined integration variables exhibit the "reality condition" (the representative of T_c symmetry in the present construction)

$$\bar{\Psi} = \Psi^T \tau, \tag{17}$$

where the matrix τ is defined through

$$\tau \equiv E^{11}_{\rm BF} \otimes \sigma^1_{\rm T} + E^{22}_{\rm BF} \otimes (i\sigma^2_{\rm T}), \tag{18}$$

and matrices with subscript "T" act in the two–component "time reversal" space introduced in (16). Armed with these definitions, the partition function may now be rewritten as

$$\mathcal{Z} = \int d\Psi \left\langle e^{-\bar{\Psi}\Psi - 2\bar{\Psi}_{\uparrow +} e^{i\left(-\omega_0 + \frac{\omega + i\delta}{4}\right)} \hat{V}^+ \Psi_{\downarrow +} - 2\bar{\Psi}_{\downarrow -} e^{i\left(\omega_0 + \frac{\omega + i\delta}{4}\right)} \hat{V}^{-T} \Psi_{\uparrow -}} \right\rangle_{\omega_0}.$$

The next step in the construction of the field theory is the average over the phase ω_0. We will perform this average with the help of the "color–flavor" transformation [21], an exact integral transform stating that for arbitrary supervectors, $\Psi_1, \Psi_2, \Psi_{1'}, \Psi_{2'}$,

$$\left\langle e^{\Psi_1^T e^{i\omega_0}\Psi_{2'}+\Psi_2^T e^{-i\omega_0}\Psi_{1'}}\right\rangle_{\omega_0} = \int d\mu(\tilde{Z},Z)\, e^{\Psi_1^T Z\Psi_{1'}+\Psi_2^T\tilde{Z}\Psi_{2'}},$$

where integration domain and algebraic structure of the supermatrices Z and $\tilde{Z}$ will be discussed in a moment.

Applied to the present context, the transformation obtains

$$\mathcal{Z} = \int d\mu(\tilde{Z},Z) \int d\Psi e^{-(\bar{\Psi}_\uparrow\Psi_\uparrow+\bar{\Psi}_\downarrow\Psi_\downarrow)+2\bar{\Psi}_{\uparrow+}\tilde{Z}\Psi_{\uparrow-}+2e^{\frac{i\omega}{2}}\bar{\Psi}_{\downarrow-}\hat{V}^{-T}Z\hat{V}^{+}\Psi_{\downarrow+}}, \tag{19}$$

where the supermatrices $Z = \{Z_{\alpha tn,\alpha' t' n'}\}$ are subject to the (convergence generating) condition $\tilde{Z}_{\alpha\alpha} = (-)^{\alpha-1}Z^\dagger_{\alpha\alpha}$. The integration measure is given by

$$\begin{aligned} d\mu(Z,\tilde{Z}) &\equiv d(Z,\tilde{Z})\mathrm{sdet}(1-Z\tilde{Z}) \\ &= d(Z,\tilde{Z})\exp\mathrm{str}\ln(1-Z\tilde{Z}), \end{aligned}$$

where "sdet" and "str" are superdeterminant and supertrace, respectively [22]. Notice that these operations include the angular momentum indices of the theory. Finally, the integration domain in the boson–boson sector is restricted by the constraint $|Z_{11}Z_{11}^\dagger| < 1$ (indices in the BF–sector.)

Using Eq. (17), the Gaussian integration over variables Ψ may be performed to obtain

$$\mathcal{Z} = \int d(Z,\tilde{Z})\, e^{\mathrm{str}\,\ln(1-Z\tilde{Z})} e^{-\frac{1}{2}\mathrm{str}\,\ln(1-\tilde{Z}\tau^{-1}\tilde{Z}^T\tau)-\frac{1}{2}\mathrm{str}\,\ln(1-e^{i\omega}\hat{U}^\dagger Z\hat{U}\tau^{-1}Z^T\tau)}. \tag{20}$$

Both the integration measure, and the action of this functional integral are invariant under transformations

$$Z \to e^{\frac{i\tilde{h}\hat{n}^2}{4}}Ze^{-\frac{i\tilde{h}\hat{n}^2}{4}},\ \tilde{Z} \to e^{\frac{i\tilde{h}\hat{n}^2}{4}}\tilde{Z}e^{-\frac{i\tilde{h}\hat{n}^2}{4}}.$$

We may use this freedom to transform the Floquet operator by $e^{\frac{i\tilde{h}\hat{n}^2}{4}}$ to the form

$$\hat{U} \equiv \exp\left(\frac{i\tilde{h}\hat{n}^2}{2}\right)\exp\left(\frac{iK\cos\hat{\theta}}{\tilde{h}}\right),$$

which will turn out to be convenient in the following. To keep the notation simple, we denote the unitarily transformed operator again by $\hat{U}$.

So far, all operations have been exact. In the rest of the paper, we aim to reduce the functional integral to a more manageable form describing long–ranged correlations in the system. To this end, we need to identify field configurations $(Z,\tilde{Z})$ of low action. Field configurations of this type necessarily have to obey the constraint

$$\tilde{Z} = \tau^{-1}Z^T\tau. \tag{21}$$

The reason is that any violation of this symmetry leads to huge mismatch between the three terms in the exponent of Eq. (20) and thereby to large values of the action.

Let us briefly discuss the conceptual status of the symmetry relation (21). The symmetry essentially involves the T–degrees of freedom of the theory. Retracing their origin, we notice that the T–indices have been introduced to account for the symmetry of the operator (2) under transposition – the physical T_c–symmetry. The ensuing doubling of indices of Z fields, and the corresponding symmetry (21) lead to the appearance of two distinct low-energy field configurations in the theory. Formally, these are the field components diagonal and off-diagonal in T–space, respectively. Physically, these fields have a status analogous to the diffusion and Cooperon modes in disordered metals.

The partition function reduced to field configurations obeying (21) is given by

$$\begin{aligned} \mathcal{Z} &= \int dZ\, e^{-S[Z]}, \\ S[Z] &= -\frac{1}{2}\mathrm{str}\,\ln(1-Z\tilde{Z}) + \frac{1}{2}\mathrm{str}\,\ln(1-e^{i\omega}\hat{U}^\dagger Z\hat{U}\tilde{Z}). \end{aligned} \tag{22}$$

We next turn to the discussion of the second fundamental symmetry of the problem, the symmetry under q–translation in angular momentum space, $\hat{T}_q$. The presence of this symmetry suggests to use a basis of Bloch functions as a preferential basis. A wave function in angular momentum space would thus be expanded as

$$\psi_n = \int_b (d\theta)\, e^{i\theta n}\, \tilde{\psi}_{n,\theta},$$
$$\tilde{\psi}_{n,\theta} = \sum_{j=-\infty}^{\infty} e^{-i\theta(qj+n)}\psi_n, \tag{23}$$

where $\int_b (d\theta) \equiv \frac{q}{2\pi}\int_0^{2\pi/q} d\theta$ and the states $\psi_{n,\theta}$ are q–periodic, $\tilde{\psi}_{n+q,\theta} = \tilde{\psi}_{n,\theta}$. This representation defines an analogous expansion of the fields Z:

$$Z_{n,a;n',a'} = \int_b (d\theta)(d\theta')\, e^{i\theta n - i\theta' n'} Z_{n,\theta,a;n',\theta',a'},$$
$$Z_{n,\theta,a;n',\theta',a'} = \sum_{jj'} e^{-i\theta(qj+n)+i\theta'(qj'+n')} Z_{n,a;n',a'}, \tag{24}$$

where $a \equiv (\alpha, t)$ is a container index comprising BF- and T-index, and the fields $Z_{n,\theta,a;n',\theta',a'}$ are q-periodic,

$$Z_{n+q,\theta,a;n',\theta',a'} = Z_{n,\theta,a;n'+q,\theta',a'} = Z_{n,\theta,a;n',\theta',a'}.$$

It may be convenient to think of the fields $Z_{n,\theta;n',\theta'}$ (we suppress indices in the notation whenever possible) as matrices in a tensor space $\mathbb{R}^q \otimes \mathcal{H}_{\rm B}$, where $\mathbb{R}^q$ is q–dimensional reduced angular momentum space (the 'unit cell'), and $\mathcal{H}_{\rm B}$ is spanned by Bloch angular wave functions $\psi(\theta)$, also with periodic boundary conditions $\psi(\theta) = \psi(\theta + 2\pi/q)$. Substituting the above expansion into the action (22) and using Eq. (3), it is straightforward to verify that the action assumes the form

$$S[Z] = -\frac{1}{2}\text{str}\ \ln(1 - Z\tilde{Z}) + \frac{1}{2}\text{str}\ \ln(1 - e^{i\omega}\hat{U}^\dagger_{\hat{\theta}} Z \hat{U}_{\hat{\theta}} \tilde{Z}), \tag{25}$$

where the supertrace now extends over the Bloch quantum numbers,

$$\text{str}(\hat{X}) = \sum_{n=1}^{q} \int_b (d\theta)\, \text{str}(X_{n,\theta;n,\theta}),$$

and the residual "str" traces over the internal indices of $\hat{X}$. Further, $\hat{U}_{\hat{\theta}} = \{U(\theta)_{n,n'}\delta(\theta - \theta')\}$ is an effective Floquet operator, diagonal in $\mathcal{H}_{\rm B}$, with matrix elements

$$U_{nn'}(\theta) = e^{i\tilde{h}\frac{n^2}{2}} \frac{1}{q} \sum_{k=0}^{q-1} e^{i\frac{K}{\tilde{h}}\cos(\frac{2\pi}{q}k+\theta) + i(n'-n)\frac{2\pi k}{q}}. \tag{26}$$

It is instructive to compare this with the matrix elements of the original Floquet operator (defined in unrestricted n-space):

$$U_{nn'} = e^{i\tilde{h}\frac{n^2}{2}} \int_0^{2\pi} \frac{d\tilde{\theta}}{2\pi}\, e^{i\frac{K}{\tilde{h}}\cos(\tilde{\theta}) + i(n'-n)\tilde{\theta}}.$$

The difference is that the compactification of the theory to a unit cell $\{0, \ldots, q\}$ in n–space implies a discretization of the angular integration, $\int \frac{d\theta}{2\pi} \to \frac{1}{q}\sum_k$. Second, the quantized "momenta" $2\pi k/q$ get shifted by the Bloch momentum $\theta \in [0, 2\pi/q]$. One may think of $U(\theta)_{nn'}$ as an effective Floquet operator on a ring (in n–space) of circumference q which is threaded by an Aharonov–Bohm flux θ. (Of course, this flux is purely fictitious and does not break time reversal invariance.) It is straightforward to verify the unitarity of the effective Floquet operator,

$$\sum_{n''} U(\theta)_{nn''}(U(\theta))^\dagger_{n''n'} = \delta_{nn'},$$

where angular momentum sums $\sum_n \equiv \sum_{n=1}^q$ are now all meant to run over the unit cell. Finally, notice that the action (25) for the field basis introduced in (24) defines a faithful representation of the theory (22); there are no approximations involved.

Let us close this section with a brief discussion of the general structure of the theory. To this end, we define the matrix

$$Q \equiv g\sigma^3_{\rm AR}g^{-1}\,, \qquad g \equiv \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}. \tag{27}$$

With the formal representation $\hat{U} \equiv \exp(i\hat{G})$, the zero frequency action can be represented as (cf. Appendix A)

$$S[Q]|_{\omega=0} = \frac{1}{2}{\rm str}\ln\left[i\tan\left(\frac{\hat{G}}{2}\right)Q + 1\right]. \tag{28}$$

This action is manifestly invariant under transformations $Q \to g_0^{-1}Qg_0$ provided that $[g_0, \hat{G}] = 0$ (or, equivalently, $[g_0, \hat{U}] = 0$). The g_0's fulfilling this condition are given by

$$g_0 = \begin{pmatrix} 1 & Z_0 \\ \tilde{Z}_0 & 1 \end{pmatrix}, \tag{29}$$

where $Z_0 \equiv \{Z_{0,\alpha t,\alpha' t'}\delta_{nn'}\}$ are unit matrices in angular momentum space. The zero-mode configurations (27) define the field space of the nonlinear σ–model (of orthogonal symmetry) [15]. The global invariance under the "zero-mode" fluctuations g_0 spanning this manifold bears important consequences for the formulation of the fluctuation expansion. It implies that we may organize our later analysis of general fluctuations (fluctuations non-commutative with $\hat{U}$) around any zero mode reference configuration Q_0. Below, we will expand around $g_0 = 0$, or $Q = \sigma^3_{\rm AR}$, corresponding to $Z_0 = 0$. In the end of the calculation, the extension to generic Q–field configurations may then be obtained by generalization of the Gaussian fluctuation action to an action displaying the full g_0 rotation invariance.

For later reference, we finally note (cf. Appendix B) that the Q-matrix representation of the density correlation function $K_\omega(n_1, n_2)$ is given by

$$K_\omega(n_1,n_2) = -\frac{1}{4}\int_b (d\theta_+)(d\theta_-)\, e^{i(n_2-n_1)(\theta_+-\theta_-)}\left\langle {\rm str}(Q_{n_2\theta_+,n_2\theta_-}\, E^{21}_{\rm AR}\otimes\mathcal{P}){\rm str}(Q_{n_1\theta_-,n_1\theta_+}\, E^{12}_{\rm AR}\otimes\mathcal{P})\right\rangle, \tag{30}$$

where $\mathcal{P} \equiv E^{11}_{\rm T}\otimes E^{11}_{\rm BF}$, the average is over the functional with action (25) and the representation (27) is implied. Note that the fields Q, by definition, are also q-periodic, i.e., $Q_{n\theta,n\theta'} = Q_{n+q\theta,n+q\theta'}$, and in Eq. (30) $|n_1 - n_2|$ may be larger than q.

IV. EFFECTIVE ACTION

In this section we reduce the functional integral formulation (22) (or, equivalently, (25)) to a more manageable field theory of localization in angular momentum space. In doing so, we need to pay attention to fluctuations $\tilde{Z}_{nn'}$ inhomogeneous in angular momentum space. We will begin by looking at the zero frequency action, $\omega = 0$, which describes the physics of infinitely long–lived correlations.

Conceptually, the field $\tilde{Z}_{nn'}$ describes the dynamical evolution of states $|n\rangle\otimes\langle n'|$. (For example, the action (22) essentially measures the overlap of the one–step evolved state, $\hat{U}\tilde{Z}\hat{U}^\dagger \leftrightarrow \hat{U}|n\rangle\otimes\langle n'|\hat{U}^\dagger$ with the un-evolved configuration.) Configurations of low dynamical action are to be expected if (a) n is close to n' (in which case, $|n\rangle\otimes\langle n'|$ evolves close to a "classical trajectory" through angular momentum space, and (b) variations in n are shallow, such that the classical action of the reference trajectory is low. Indeed, in the limiting case $Z_{nn'} \simeq B_n\delta_{nn'}$ and B_n independent of n, the operators $\hat{U}$ and $\hat{U}^\dagger$ in (22) cancel out and the action vanishes. Our goal is to compute the action associated to soft fluctuations around this zero mode limit.

A. Fluctuation action

As discussed in the end of the foregoing section, it will be sufficient to analyze angular momentum space fluctuations around the reference point $Z = 0$. The structure of the full action then follows from its invariance under uniform rotations g_0. We thus start by considering the quadratic expansion of the action (25) around $Z = 0$ and at $\omega = 0$. Denoting the ensuing quadratic action by $S_{\rm fl}$, we have

$$\begin{aligned} S_{\rm fl}[Z] &= \frac{1}{2}{\rm str}(\tilde{Z}Z - \tilde{Z}\hat{U}^\dagger Z\hat{U}) \qquad (31)\\ &= \frac{1}{2}\sum_{nn'n''n'''}\int_b (d\theta)(d\theta')\,{\rm str}(Z_{n\theta,n''\theta'}\tilde{Z}_{n'\theta',n'''\theta})(\delta_{n,n'''}\delta_{n',n''} - U(\theta')_{n'',n'}U(\theta)^\dagger_{n''',n}). \end{aligned}$$

This is an exact representation of the quadratic fluctuation action. We now need to identify those configurations Z whose action is asymptotically vanishing. All other, "massive" field configurations can then be integrated out by perturbation theory. Following our discussion in the end of the previous section, we split the fields as

$$Z = B + C$$

into "massless" configurations B and "massive" fluctuations C. The massless sector will be parameterized as

$$B_{n\theta,n'\theta'} = q^{-1} \sum_{\phi} e^{-i\phi n} B_{\theta,\theta'}(\phi)\, \delta_{nn'}, \tag{32}$$

where $\sum_\phi$ is a shorthand for a summation over "mode indices" $\phi = 0, 2\pi/q, 4\pi/q, \ldots, \phi_0$ smaller than a certain cutoff index ϕ_0 to be discussed in a moment. The fields B describe fluctuations diagonal in angular momentum space, $n = n'$. To understand the meaning of this limitation, recall the interpretation of $Z_{n_1 n_2}$ as the representative of a bilinear $|n_1\rangle\langle n_2|$. Turning to a Wigner representation (symbolic notation), $|n_1\rangle\langle n_2| \to W(n,\theta) \equiv \sum_{\Delta n} e^{i\Delta n\theta} |n + \Delta n/2\rangle\langle n - \Delta n/2|$, we notice that the degree of off-diagonality $\Delta n = n_1 - n_2$ encodes information about the direction of propagation in angular momentum space (as measured by the angular variable θ of the rotating particle.) On physical grounds, we expect that sense of direction to decay rapidly if the kicking strength K is sufficiently large (for fixed $\tilde{h}$). Indeed we will see in Appendix C that the effect of off–diagonal fluctuations $C_{n,n'}$ amounts to largely innocent $\sim K^{-1}$ corrections to the theory. The essential information is carried by the modes $B_{n,n}$ describing fluctuations uniform in angular space. The Fourier mode decomposition in terms of an index $\phi < \phi_0$ states that we will concentrate on fluctuations at scales $n \gtrsim 2\pi/\phi_0$. Deferring the self-consistent determination of ϕ_0 to Appendix C we here state that $\phi_0 \ll \tilde{h}/K = \ell^{-1}$, i.e. fields of low action will fluctuate on scales $\gg \ell$, where the scale ℓ plays a role analogous to that of the elastic mean free path in disordered systems. It is important that the cutoff index ϕ_0 is invariant under the transformation (6). (Note also that the condition $2\pi/q < \phi_0 \ll \tilde{h}/K$ implies $q \gg \ell$, as we assume above. Otherwise, the massless sector will contain only the constant mode $B_{n,n} = \text{const.}$)

For the moment we ignore the contribution of the massive modes and substitute $Z \simeq B$, with B parameterized as in (32) into the action. This leads to

$$S_{\text{fl}}[B] = \frac{1}{2}\frac{1}{q^2} \sum_{\phi,\phi'} \int_b (d\theta)(d\theta')\text{str}(B_{\theta,\theta'}(\phi)\tilde{B}_{\theta',\theta}(\phi'))K_{\theta,\theta'}(\phi,\phi'),$$

where the integral kernel is given by

$$K_{\theta,\theta'}(\phi,\phi') = \sum_{nn'} (\delta_{nn'} - U_{nn'}(\theta')U^\dagger_{n'n}(\theta))e^{-i\phi n - i\phi' n'}.$$

Substituting (26), it is a straightforward calculation to reduce this expression to

$$\begin{aligned} K_{\theta,\theta'}(\phi,\phi') &= \left[q - \sum_{k=0}^{q-1} e^{-i\frac{K}{\tilde{h}}\sin\left(\frac{2\pi k}{q}(\theta-\theta'+\phi)\right)} \right] \delta_{\phi,-\phi'} \\ &= q\left(\frac{K}{2\tilde{h}}\right)^2 (\theta-\theta'+\phi)^2 \delta_{\phi,-\phi'} + \mathcal{O}(K\phi/\tilde{h})^4. \end{aligned}$$

In the last expression, we have approximated the discrete k–sum by an integral. (For the small values of ϕ, θ, θ' under consideration, this is an innocent operation.) Substituting this result into the action and switching back to a momentum space representation $B(\phi) \to B(n)$, we obtain

$$S_{\text{fl}}[B] = \frac{D_0}{2} \int_0^q dn\, \text{str}((-i\partial_n + [\hat{\theta},\ ])B(+i\partial_n - [\hat{\theta},\ ])\tilde{B}),$$

where we have passed from sum to integration because $B(n)$ is interpreted as a smooth function of the variable n. Here, we have introduced the (bare) diffusion constant

$$D_0 = \left(\frac{K}{2\tilde{h}}\right)^2, \tag{33}$$

and the commutators act as

$$([\hat{\theta}, B])_{\theta,\theta'} = (\theta - \theta')B_{\theta,\theta'}.$$

The action above was obtained by quadratic expansion around the reference configuration $Q = \sigma^3_{\rm AR}$ (cf. discussion in the end of Section III.) To obtain its rotationally invariant generalization, we rewrite the action as

$$S_{\rm fl}[Q] = -\frac{D_0}{16}\int_0^q dn\,{\rm str}((-i\partial_n + [\hat\theta,\ ])Q(i\partial_n - [\hat\theta,\ ])Q), \tag{34}$$

where $Q = \tilde g \sigma^3_{\rm AR}\tilde g^{-1}$ and $\tilde g = \begin{pmatrix} 1 & B \\ \tilde B & 1\end{pmatrix}$ and $\tilde g^{-1} = \begin{pmatrix} 1 & -B \\ -\tilde B & 1\end{pmatrix}$ describe infinitesimally small deviations off $\sigma^3_{\rm AR}$. The generalization to an arbitrary configuration Q is then given by $\tilde g \to g$ and $Q \to Q \equiv g\sigma^3_{\rm AR}g^{-1}$, where g need no longer be close to unity. We thus conclude that action (34) defined for arbitrary

$$Q \equiv g\sigma^3_{\rm AR}g^{-1}\,, \qquad g \equiv \begin{pmatrix} 1 & B \\ \tilde B & 1\end{pmatrix}, \tag{35}$$

is the unique rotationally invariant generalization of the quadratic action above. (Starting from (25), the same action can be obtained by the straightforward if more tedious fluctuation expansion around arbitrary Z.)

To conclude the derivation of Eq. (34), we need to explore its stability with respect to the "massive mode" fluctuations C that were so far neglected. This analysis is conceptually straightforward yet tedious in practice. Referring to Appendix C for the actual formulation of the massive mode integration, we here summarize the main conclusions:

- The soft mode action is stable at length scales $\Delta n \gtrsim \ell = \frac{K}{\tilde h}$.
- Massive mode fluctuations are damped out at scales $< \ell$.
- Their principal feedback into the soft mode action is a weak renormalization of the diffusion constant, $D_0 \to D_q$, where the so–called quantum diffusion constant [8, 20] is given by

  $$D_q \simeq D_0(1 - 2J_2\,(K_q) - 2J_1^2\,(K_q) + 2J_3^2\,(K_q)) + \ldots \tag{36}$$

  with

  $$K_q \equiv \frac{2K}{\tilde h}\sin\left(\frac{\tilde h}{2}\right), \tag{37}$$

  and $J_n(x)$ are the Bessel functions of the first kind (cf. Eq. (C16).)
- At certain irrational values [23], $\tilde h$ assumes the form of an infinite fraction. At these values, long lived angular correlations form, and we are not able to get the massive mode fluctuations under control. At these values, the theory considered in this paper does not apply.

B. Frequency action

To complete the derivation of the effective action, we need to take the finiteness of the control parameter ω into account. This will generate a second contribution to the action, $S_\omega[Q]$, adding to the fluctuation action above. In deriving $S_\omega[Q]$, we will resort to two (parametrically controlled) approximations: first, we will ignore the massive modes in the computation of the finite frequency action [24, 25]. Second, we will ignore operators of $\mathcal{O}(\omega\partial_n^2)$. At large spatial scales, $\partial_n < \tilde h/K$ and low frequencies, $\omega \ll 1$, the neglected terms are irrelevant. Technically, the assumptions above imply that in the expansion of the action (25) to lowest order in ω, we will set $Z \simeq B$, and ignore terms arising from the non-commutativity of Z and $\hat U$ (which would generate contributions of $\mathcal{O}(\omega\partial_n^2)$.)

We thus start out from

$$\begin{aligned} S_\omega[B] &\simeq -\frac{1}{2}{\rm str}\,\ln(1 - B\tilde B) + \frac{1}{2}{\rm str}\,\ln(1 - e^{i\omega}B\tilde B) \\ &\simeq -\frac{i\omega}{2}{\rm str}\left(\frac{B\tilde B}{1 - B\tilde B}\right). \end{aligned}$$

Comparison with the definition (35) shows that this can be rewritten as

$$S_\omega[Q] = -\frac{i\omega}{8}{\rm str}\left(Q\sigma^3_{\rm AR}\right).$$

Combining this with the fluctuation action (34), we arrive at the main result of this paper, the invariant soft mode action,

$$S[Q] = \frac{1}{16}\int_0^q dn\,\mathrm{str}\left(D_q(i\partial_n - [\hat{\theta},\ ])Q(i\partial_n - [\hat{\theta},\ ])Q - 2i\omega Q\sigma^3_{\mathrm{AR}}\right). \tag{38}$$

Notice that the action above is universal in that it depends on system parameters only through the diffusion constant $D_q = D_q(K, \tilde{h})$. The diffusion constant, D_q, and therefore the theory as a whole, are invariant under the transformation $(\tilde{h}, K) \to (\tilde{h}_\epsilon, K_\epsilon)$. (Note that, for $K_\epsilon \lesssim 1$, the constructions discussed in this paper do not work.)

Conceptually, $S[Q]$ describes the physics of a disordered metallic *ring* (the fields Q obey periodic boundary conditions $Q(n) = Q(n+q)$) subject to a fictitious magnetic flux θ. (Indeed, the action is mathematically similar to the nonlinear σ-model action of persistent current system, a disordered metallic ring threaded by an Aharonov–Bohm flux [15]). A magnetic flux through the ring acts on Hilbert space states as $|n\rangle\langle n'| \to e^{i\theta n}|n\rangle\langle n'|e^{-i\theta n'}$, and the corresponding "covariant derivative" governs the action (38). However, the Aharonov–Bohm analogy must not be carried too far: the operator $[\hat{\theta},\ ]$ does not represent a genuine magnetic flux. Rather, it carries the information on the infinite extension of the "real" system, which is different from the single unit cell system for which (38) is defined. The difference to a real flux is that we are integrating over all gauge sectors (corresponding to field configurations $B_{\theta,\theta'}$ for all values θ and θ') at once, and this restores the time reversal invariance of the system (which in the presence of a real flux would be lost.) These differences notwithstanding, the Aharonov–Bohm analogy will be a useful guiding principle in the discussion of the system below.

Let us conclude this section with a general observation that will simplify our discussion below: in principle, the degrees of freedom of the theory are "infinite dimensional matrices", $\{B_{\theta,\theta'}\}$ depending on the continuous index θ. However in the absence of sources such as those specified in (30), the integration over this matrix field gives unity, $\mathcal{Z} = 1$. This is a consequence of supersymmetry. In the presence of a symmetry breaking source (the projector $\mathcal{P}$ in (30)), supersymmetry gets *partially* lost. Specifically, the decoupling of the source from all θ–indices other than $\theta_\pm$ in (30) implies that the integration over the variables $B_{\theta,\theta'}$, $(\theta,\theta') \neq (\theta_+,\theta_-)$ does not contribute to the functional integral. These variables may simply be removed from the integration manifold (as long as we are probing a functional expectation value with fixed Bloch index configuration as in (30).) This reduction is a rigorous consequence of supersymmetry. In effect, it implies that the continuous index set $\{(\theta,\theta')\}$ gets reduced to one element (θ_+,θ_-), i.e. $B_{\theta,\theta'} \to B_{\theta_+,\theta_-}\delta_{\theta,\theta+}\delta_{\theta',\theta_-}$. Since $\theta_\pm$ are fixed, we need not keep them as indices, i.e. $B_{n\theta_+ a, n\theta_- a'} \to B_{na,na'}$. Similarly, $Q_{a\theta,a'\theta'}(n) \to Q_{a,a'}(n)$, where the reduced Q is generated by the reduced B. In the effective action (38), the reduced form of the operator $\hat{\theta}$ reads

$$\hat{\theta} = (\theta_+ E^{11}_{\mathrm{AR}} + \theta_- E^{22}_{\mathrm{AR}}) \otimes \sigma^3_{\mathrm{T}}, \tag{39}$$

where the matrix σ^3_{T} represents the sign change in the Bloch "flux" under time reversal. Throughout, we will mostly work in the reduced representation.

V. OFF RESONANCE

In the next two sections, we will employ the effective field theory (38) to discuss the long-ranged physics of the system. Our discussion is organized in two parts: we first discuss the case of (near) irrational $\tilde{h}/4\pi$ where $\xi/q \to 0$ and $\xi \sim D_q$ will be the localization length of the system. In this regime, periodicity effects relating to the q–commensurability of Planck's constant play no role and the system can be regarded as infinitely extended in angular momentum space. In a manner to be reviewed below, it then becomes similar to an infinitely extended disordered wire. This limit will serve as a benchmark against which to compare the physics of the q-periodic system.

The commutators $[\hat{\theta},\ ]$ appearing in the effective action have an eigenvalue spectrum limited by $2\pi/q$. For $\xi/q \to 0$, this is negligible in comparison to the excitation gap $\sim \xi^{-1}$ of the localized systems, and we may simply ignore the θ–contribution to the effective action.

In this regime the effective action (38) reduces to the nonlinear σ-model action of quasi one–dimensional disordered metals [15, 26]. To discuss the localization behavior of this system, we consider the density–density correlation function defined in (8). We first note that the system is "translationally invariant" in angular momentum space, meaning that $K_\omega(n_1, n_2) = K_\omega(n_1 - n_2)$ depends only on coordinate differences. As such, it can be expressed in a Fourier representation, $K_\omega(\phi) \equiv \sum_n e^{-i\phi n} K_\omega(n)$. Comparison with Eq. (10) shows that this function yields the energy increase as

$$E(\tilde{t}) = -\frac{1}{2}\int \frac{d\omega}{2\pi} e^{-i\omega\tilde{t}} \partial^2_\phi\big|_{\phi=0} K_\omega(\phi). \tag{40}$$

By unitarity, the function K has to obey the limit behavior $\lim_{\phi\to 0} K_\omega(\phi) = \frac{i}{\omega}$. The most general low ϕ asymptotic compatible with this requirement (and time reversal symmetry) reads

$$K_\omega(\phi) = \frac{1}{-i\omega + D(\omega)\phi^2}, \tag{41}$$

where $D(\omega)$ is the so–called dynamical diffusion constant.

Referring for a much more extensive discussion to Ref. [15], we here briefly discuss the behavior of $D(\omega)$ upon lowering the frequency:

- At high frequencies, $\omega D_q \to \infty$, fluctuations of the fields Q are strongly damped and the effective action (38) may be approximated by the quadratic expansion around $Q = \sigma^3_{\mathrm{AR}}$:

  $$S[B] \simeq \frac{1}{2}\int dn\, \mathrm{str}(B(-D_q\partial_n^2 - i\omega)\tilde{B}). \tag{42}$$

  Likewise, the correlation function (30) reduces to its Gaussian representation

  $$K_\omega(n_1, n_2) = \langle \mathrm{str}(B(n_2)\,\mathcal{P})\,\mathrm{str}(\tilde{B}(n_1)\,\mathcal{P})\rangle. \tag{43}$$

  Doing the Gaussian integral, we obtain (41) with $D(\omega) = D_q$ given by its bare value. Substitution into (40) shows that the kinetic energy increases as

  $$E(\tilde{t}) = D_q\tilde{t}, \tag{44}$$

  i.e. the variable $E \sim n^2$ shows the behavior characteristic of diffusive dynamics.

- Upon lowering ω, but still keeping $\omega > D_q^{-1}$, corrections to the bare value begin to form. The ensuing perturbation series is controlled by the parameter $(\omega D_q)^{-1}$. Technically, these corrections may be obtained by expansion of the action (38) to higher order in the constituent fields B and subsequent integration against the Gaussian kernel (42). To leading order, this leads to a renormalization of the diffusion constant,

  $$D(\omega) = D_q\left[1 - 2\int \frac{d\phi}{2\pi}\frac{1}{-i\omega + D_q\phi^2}\right]. \tag{45}$$

  Physically, the correction term represents the modification of the diffusion constant by quantum weak localization corrections [12, 13]. The reduced diffusivity leads to a lowering of the energy increase, $E(\tilde{t}) = D_q\tilde{t} - \frac{4}{3\sqrt{\pi}}D_q\tilde{t}^{3/2}$ where $\tilde{t} \ll D_q$.

- Finally, at low frequencies, $\omega \ll D_q^{-1}$, the function $K_\omega(\phi)$ has the form

  $$K_\omega(\phi) \overset{\omega\ll D_q^{-1}}{\longrightarrow} -\frac{A(\phi)}{i\omega}, \qquad A(\phi) = 1 - \zeta(3)D_q^2\phi^2 + \mathcal{O}(D_q^4\phi^4), \tag{46}$$

  where ζ is the Riemann ζ function. It indicates that the diffusion constant approaches the value

  $$D(\omega) = -i\zeta(3)D_q^2\omega. \tag{47}$$

  Eq. (47) implies vanishing diffusivity at low frequencies, a hallmark of strong dynamical localization. Substitution into (40) indeed shows that

  $$E(\tilde{t}) \overset{t\gg D_q}{\sim} D_q^2, \tag{48}$$

  saturates at a constant value. At the saturation time $\tilde{t}_L \sim D_q$, the two results (44) and (48) match. Therefore, the localization length scales as $\xi \sim \sqrt{D_q\tilde{t}_L} = D_q$.

  To make the phenomenon of localization more explicit, one may consider the asymptotic form of the real time representation of the density correlator, $K(n) \equiv K(n, \tilde{t} \to \infty)$. This function is given by [15, 26]

  $$K(n) \overset{|n|\gg\xi}{\sim} \frac{1}{4\xi}\left(\frac{4\xi}{|n-n'|}\right)^{3/2} \exp\left(-\frac{|n|}{4\xi}\right), \tag{49}$$

where we have neglected the numerical prefactor and have defined

$$\xi = D_q/2 \tag{50}$$

as the localization length. This formula was first obtained by Shepelyansky [20] based on phenomenological arguments.

Summarizing, the QKR supports diffusive excitations in angular momentum space at length scales $\Delta n > K/\tilde{h} \gg 1$ before giving way to localization at scales $\Delta n > \xi \sim D_q$. These results hold for irrational values of $\tilde{h}$, and sufficiently large values of the kicking strength, K. For small values of Planck's constant, $\tilde{h} \ll 1$, the parameter K_q in (37) approaches the classical limit, $K_q \stackrel{\tilde{h}\ll 1}{\longrightarrow} K$, and the diffusion constant becomes

$$D_q \stackrel{\tilde{h}\ll 1}{\longrightarrow} \left(\frac{K^2}{2\tilde{h}}\right)^2 \left(1 - 2J_2(K) - 2J_1^2(K) + 2J_3^2(K)\right), \tag{51}$$

where the corrections are the classical Rechester-White corrections [27, 28]. In this limit, our result $\xi = D_q/2$ confirms an empirical formula by Shepelyansky and co-workers [3, 29–31].

VI. RESONANCES

We now consider the system at rational $\tilde{h} = 4\pi p/q$ [2]. Pioneering work on the physics of the kicked rotor at these "resonant" values of Planck's constant has been done by IS [6] (see also Ref. [8] for review.) To assist the orientation of the reader, we begin by reviewing some key results of that work in the language of the present formalism. We will then proceed to extend the theory so as to include the effects of diffusion and Anderson localization. (While the ramifications of localization at resonant values of $\tilde{h}$ have been addressed before [6, 7], these discussions were based on phenomenology and numerical diagonalization.) Throughout this section, it will be convenient to represent the function $E(\tilde{t})$ as (cf. Eq. (10))

$$E(\tilde{t}) = -\frac{1}{2}\int \frac{d\omega}{2\pi}\, e^{-i\omega\tilde{t}} \sum_{n=0}^{q-1} \int_b (d\theta)\partial_{\theta'}^2\big|_{\theta'=\theta} K_\omega(n), \tag{52}$$

where $K_\omega(n)$ is the density correlation function evaluated at fixed parameters θ, θ'.

A. Theory of Izrailev and Shepelyansky

In Ref. [6], the periodicity effects in angular momentum space were discussed in terms of an eigenbasis of the effective Floquet operator (26). To reproduce these results in the language of the present formalism, we start from the formal diagonalization

$$U_{nn'}(\theta) = \sum_j R_{nj}(\theta) e^{i\epsilon_j(\theta)} R^\dagger_{jn'}(\theta),$$

where $\hat{R}(\theta) \equiv \{R_{ij}(\theta)\}$ are unitary $q \times q$ matrices parametrically depending on the Bloch angle, and $\epsilon_j(\theta)$ are the θ–dependent eigenenergies. We may use the transformation $\hat{R}(\theta)$ to transform the fields Z in (31) to an "eigenbasis representation",

$$Z_{n\theta,n'\theta'} \to R_{nj}(\theta) Z_{j\theta,j'\theta'} R^\dagger_{j'n'}(\theta').$$

This transformation is unitary and does not affect the integration measure. As a result the Gaussian expansion of the action assumes the form

$$S[Z] = \frac{1}{2}\sum_{jj'} \int_b (d\theta)(d\theta')\, \mathrm{str}(Z_{j\theta,j'\theta'} \tilde{Z}_{j'\theta',j\theta}) \left(1 - e^{i(\epsilon_{j'}(\theta') - \epsilon_j(\theta) + \omega)}\right),$$

where we have included the frequency contribution to the quadratic action. Likewise, the Gaussian approximation to the density correlation function (30) is given by

$$\begin{aligned} K_\omega(n_1,n_2) &= \int_b (d\theta)(d\theta')\, e^{i(n_2-n_1)(\theta-\theta')}(R_{n_2,j}(\theta)R^\dagger_{n_2,j'}(\theta'))\,(R_{n_1,j'}(\theta')R^\dagger_{n_1,j}(\theta))\left\langle \mathrm{str}(Z_{j\theta,j'\theta'}\mathcal{P})\,\mathrm{str}(\tilde{Z}_{j'\theta',j\theta}\mathcal{P})\right\rangle \\ &= \int_b (d\theta)(d\theta')\, e^{i(n_2-n_1)(\theta-\theta')}\frac{(R_{n_2,j}(\theta)R^\dagger_{n_2,j'}(\theta'))\,(R_{n_1,j'}(\theta')R^\dagger_{n_1,j}(\theta))}{1-e^{i(\epsilon_{j'}(\theta')-\epsilon_j(\theta)+\omega)}}, \end{aligned}$$

where in the last step we did the Gaussian integral over the action $S[Z]$. This result is but a formal expansion of the density-density correlation function in a Bloch eigenbasis [6]. It may be employed to obtain estimates for the energy increase $E(\tilde{t})$, as given by Eq. (10):

$$E(\tilde{t}) = -\frac{1}{2}\sum_{n=0}^{q-1}\int_b (d\theta)\partial^2_{\theta'}\big|_{\theta=\theta'}\{(e^{i(\epsilon_{j'}(\theta')-\epsilon_j(\theta))\tilde{t}}-1)(R_{n,j}(\theta)R^\dagger_{n,j'}(\theta'))\,(R_{0,j'}(\theta')R^\dagger_{0,j}(\theta))\}. \tag{53}$$

This formula has been derived by IS [6]. Except for a few small values of $q(=1,2,4)$, it cannot be evaluated in closed terms. However, the general structure of the result indicates that the energy stored in the system increases as in Eq. (4). Specifically, for large times the energy increases quadratically. To understand this asymptotic behavior, notice that at large times $\tilde{t} \gg \Delta^{-1}$, where the level spacing $\Delta \sim |\epsilon_j - \epsilon_{j+1}|$ is a measure of the typical separation between neighboring levels, the dominant contribution to (53) comes from the diagonal term, $j = j'$. Describing the auto–correlation of individual levels (or "bands"), $\epsilon_j(\theta)$, this term is of "deep quantum origin". In this limit, the θ-derivatives dominantly act on the rapidly oscillatory phases $\sim \exp(i\epsilon_j(\theta')\tilde{t})$, and we arrive at the estimate

$$\eta \simeq \frac{1}{2}\sum_{n=0}^{q-1}\int_b (d\theta)|R_{n,j}(\theta)|^2\,|R_{0,j}(\theta)|^2(\partial_\theta\epsilon_j(\theta))^2.$$

Notice that the sum $\sum_{n=0}^{q-1}\int_b (d\theta)|R_{n,j}(\theta)|^2\,|R_{0,j}(\theta)|^2$ is (approximately) normalized to unity. This means that

$$\eta \simeq \frac{1}{2}\langle(\partial_\theta(\epsilon_j(\theta)))^2\rangle, \tag{54}$$

can be interpreted as measure of the typical "level velocity" corresponding to the Bloch–phase dependence, and the average is taken over different quasi-energy levels (for fixed Bloch angle θ). Corrections to the dominantly quadratic increase arise from other derivative combinations, where the time derivative acts once (the linear term in (4)) or twice (the term $b(\tilde{t})$) on the diagonalizing matrices.

However, at shorter times $\tilde{t} < \Delta^{-1}$, inter–band contributions $j \neq j'$ begin to matter, and these can not be effectively described by a formal eigenfunction decomposition. In the next section, we present an analytical theory of resonances in the regime $\ell < q < \xi$ which yields the full time dependent profile $E(\tilde{t})$. In section VI C, we will briefly address the regime of vary large periodicity intervals, $q > \xi$.

B. Resonances in the regime $\ell < q < \xi$

We will now address the physics of resonances within the framework of the theory (38). Specifically, we will aim to obtain concrete expressions for the energy increase $E(\tilde{t})$ in terms of universal system parameters.

1. *Frequencies* $\omega \gg \Delta$

We start by considering the regime of frequencies $\omega \gg \Delta$. Recall that $\Delta = 2\pi/q$ is the effective quasi–energy level spacing (or the Bloch "band gap") of a periodicity volume. Thanks to the large value of ω, we may expand Q to quadratic order in B and consider the Gaussian approximation to the action, (42), where, however, the derivatives $\partial_n \to \partial_n + i[\hat{\theta},\ ]$ have to be generalized to covariant derivatives and the commutator acts as

$$[\hat{\theta}, B] = \theta_+\sigma^3_{\mathrm{T}}B - \theta_- B\sigma^3_{\mathrm{T}}.$$

Switching to a Fourier representation $B(n) = q^{-1}\sum_\phi e^{-i\phi n}B(n)$ and doing the Gaussian integral against the source terms (43), we obtain for $K_\omega(n) \equiv K_\omega(n,0)$

$$K_\omega(n) = q^{-1}\sum_\phi \int_b (d\theta)(d\theta') \frac{e^{-i(\phi+\theta-\theta')n}}{D_q(\phi+\theta-\theta')^2 - i\omega}.$$

We next substitute this result into Eq. (10) and perform the straightforward integrals over n and ω. As a result, we obtain Eq. (44); the periodicity of the system does not play a significant role here.

2. Frequencies $\omega \lesssim \Delta$

For $\omega \lesssim \Delta$ and flux parameters $\theta_+ \simeq \theta_-$ relevant to the computation of energy increase, the action of quadratic fluctuations is no longer small in comparison to unity. This signals a breakdown of perturbation theory. Indeed, we anticipate from our discussion of Section VI A that this is the regime, where the band–dispersion, $\partial_\theta \epsilon_j(\theta)$, of individual levels begins to matter. It is known [15] that the fine structure of individual levels of hyperbolic systems is encoded in non–perturbative (in ω) fluctuations of the field theory.

Technically, our setup bears similarity to the problem of "parametric correlations" [32]. There, one usually considers correlations in the density of states of a system at slightly different values of some control parameter. In the present context, the role of that parameter is taken by the Bloch phases $\theta_\pm$. (For earlier studies of Bloch–phase parameter correlations applied to the physics of periodically extended Hamiltonian chaotic systems, see Refs. [33, 34].)

To set the stage for the non–perturbative calculation, we first note that for $\omega \lesssim \Delta$, inhomogeneous fluctuations (modes of finite momentum ϕ) have a typical action $E_c/\Delta > 1$ (the action of the first non–vanishing Perron-Frobenius eigenmode of a diffusive system). This is much larger than the action of the homogeneous mode, $Q(n) \equiv Q = \text{const.}$, which means that inhomogeneous fluctuations can be neglected. Substitution of the constant configuration into (38) shows that the "zero-mode action" is given by

$$S[Q] = \frac{\pi}{8\Delta}\mathrm{str}(D_q[\hat\theta, Q]^2 - 2i\omega Q\sigma^3_{\mathrm{AR}}).$$

Referring to Ref. [15] for an in-depth discussion of the technical details, we here merely note, that the integration over Q is best performed in a polar coordinate representation of the Q-matrices. The idea is to represent Q as $Q = \mathcal{R}Q_0\mathcal{R}^{-1}$, where $[\mathcal{R}, \sigma^3_{\mathrm{AR}}] = 0$ and the "radial" matrix Q_0 is given by

$$Q_0 = \begin{pmatrix} \cos\hat\Theta & i\sin\hat\Theta \\ -i\sin\hat\Theta & \cos\hat\Theta \end{pmatrix}_{\mathrm{AR}}, \tag{55}$$

$$\hat\Theta = \begin{pmatrix} \hat\theta_{11} & 0 \\ 0 & \hat\theta_{22} \end{pmatrix}_{\mathrm{BF}}, \quad \hat\theta_{11} = \begin{pmatrix} \tilde\theta & 0 \\ 0 & \tilde\theta \end{pmatrix}_{\mathrm{T}}, \quad \hat\theta_{22} = i\begin{pmatrix} \tilde\theta_1 & \tilde\theta_2 \\ \tilde\theta_2 & \tilde\theta_1 \end{pmatrix}_{\mathrm{T}}$$

with $0 < \tilde\theta < \pi, \tilde\theta_{1,2} > 0$. The action $S[Q]$ is "rotationally invariant" in that it couples only to the radial coordinates ($\lambda \equiv \cos\tilde\theta, \lambda_{1,2} \equiv \cosh\tilde\theta_{1,2}$) of the polar decomposition. Of these three variables, $\lambda \in [-1,1]$ is "compact", while $\lambda_{1,2} \in [1,\infty]$ parameterize the non–compact domain of the super integration manifold.

Inserting the polar decomposition of Q into the zero-mode action, we find

$$S = \frac{\pi}{2\Delta}\{D_q[(\Delta\theta)^2(\lambda_1^2 - \lambda^2) + (\theta_+ + \theta_-)^2(\lambda_2^2 - 1)] + 2i\omega(\lambda - \lambda_1\lambda_2)\},$$

where $\Delta\theta \equiv \theta_+ - \theta_-$. The action S then suggests that, for generic values of the phases $\theta_\pm = \mathcal{O}(2\pi/q) = \mathcal{O}(\Delta)$, Q–fluctuations off–diagonal in T-space (characterized by λ_2) are penalized by a large action $\mathcal{O}(D_q\Delta)$. Physically, this means that for fixed values of the phases $\theta_\pm$ time reversal invariance is effectively broken (the symmetry gets restored only upon integration over all phases), and Cooperon–like fluctuations (fluctuations off-diagonal in T–space) are strongly damped. Under these circumstances, we may restrict ourselves to T–diagonal fluctuations ("diffusons") and set $\lambda_2 = 1$. The action then collapses to

$$S = \frac{\pi D_q(\Delta\theta)^2}{2\Delta}(\lambda_1^2 - \lambda^2) + i\frac{\pi\omega}{\Delta}(\lambda - \lambda_1), \tag{56}$$

and the density correlation function $K_\omega(\Delta\theta)$ is reduced to

$$K_\omega(\Delta\theta) = \frac{q}{4}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda \frac{\lambda_1 + \lambda}{\lambda_1 - \lambda} e^{-\frac{\pi D_q(\Delta\theta)^2}{2\Delta}(\lambda_1^2 - \lambda^2) + i\frac{\pi\omega}{\Delta}(\lambda_1 - \lambda)}. \tag{57}$$

To proceed further we substitute this result into Eq. (52) and perform the remaining integral exactly. As a result, we obtain

$$E(\tilde{t}) = -\frac{1}{2}\int \frac{d\omega}{2\pi}\frac{e^{-i\omega\tilde{t}}}{\omega^2}D(\omega), \tag{58}$$

where $D(\omega)$ is given by (see Fig. 2, the top and middle panel)

$$D(\omega) = D_q\left[1 - \frac{2}{q^2\omega^2}\left(1 - e^{iq\omega}\right)\right]. \tag{59}$$

This result for the frequency dependent diffusion constant in periodic structures was first obtained in Ref. [33]. Doing the integral over frequencies in Eq. (58) we obtain Eq. (5) (see Fig. 2, bottom panel).

Eq. (5) is the main result of the present section. It accurately describes the function $E(\tilde{t})$, up to corrections in $\exp(-E_c\tilde{t})$ (the effect of fluctuations inhomogeneous across the unit cell). Eq. (5) coincides (including the coefficients) with a conjecture on the mean square displacement of periodic quantum maps (mimicking one-dimensional quantum random walks) formulated by Wójcik and Dorfman [35]. That conjecture was based on the phenomenological modelling of quantum maps in terms of random matrix theory. Note that in the short time regime, $x < 1$ or $\tilde{t} < q$, the quantum correction to the leading (diffusive) terms scales as $(\tilde{t}/q)^3$. As pointed out in Ref. [35], this scaling reflects universal fluctuations of the quasi–energy spectra and eigenfunctions of the system. To the best of our knowledge, this effect has not been discussed in connection with the QKR.

Turning to the long time regime, $x > 1$ or $\tilde{t} > q$, comparison with the formal result (54) leads to the identification

$$\langle(\partial_\theta\epsilon_j(\theta))^2\rangle = \frac{2D_q}{q} \approx \frac{K^2}{q\tilde{h}^2} \tag{60}$$

of the typical level velocity square. To understand the meaning of this equation, notice that the so–called Thouless conductance [36] of the periodicity volume

$$g = \frac{E_c}{\Delta} = \frac{1}{\Delta^2}\langle(\partial_\phi\epsilon_j)^2\rangle$$

can be understood in terms of the typical sensitivity of energy levels to changes in the boundary conditions $\psi(q) = \psi(0)e^{i\phi}$. In the present context $\phi = q\theta \sim \theta/\Delta$ is set by the Bloch phase. Substituting this identification into the formula above and using that $E_c \sim D_q\Delta^2$, we obtain (60). Qualitatively, our results may be extrapolated into the regime where ℓ is comparable to q. Even quantitatively, (60) is in good agreement with the numerical finding of Ref. [6], where $\langle(\partial_\theta\epsilon_j(\theta))^2\rangle$ was found to be $0.8K^2/(q\tilde{h}^2)$.

C. Resonances in the regime $q > \xi$

Let us now briefly explore what happens in regimes where the periodicity interval exceeds the localization length. Nominally, we are still dealing with a q-periodic quantum system whose spectrum can be organized in Bloch bands. However, the fact that wave functions overlap only exponentially weakly $\mathcal{O}(\exp(-q/(8\xi)))$ across the q-volume means that these bands are exponentially narrow [8]. Accordingly, the level velocities $\partial_\theta\epsilon_j \sim \exp(-q/(8\xi))$ are exponentially small meaning that the $\mathcal{O}(\tilde{t}^2)$ increase of the energy in large volumes $q \gg \xi$ will be weighted by an exponentially small coefficient. We do not aim to analyze this correction any further.

Turning to the dominant contributions to $E(\tilde{t})$, we expect a profile as in the $q \to \infty$ system, diffusive spreading at times $\tilde{t} < D_q$ followed by saturation at larger times. To see how this comes about, notice that the principal effect of the finiteness of q is a substitution $\phi \to \phi + \Delta\theta$ in the argument of the density correlation function $K_\omega(\phi)$. As far as the function $E(\tilde{t})$ is concerned, this change plays no role, the reason being that the derivative in (40) needs to be performed at both $\phi = 0$ and $\Delta\theta = 0$. This shows that the function $E(\tilde{t})$ behaves as in the $q \to \infty$ system, apart from the exponentially small tail corrections mentioned above.

VII. CONCLUDING REMARKS

Extending earlier work [14], we have introduced a microscopic and parametrically controlled theory of quantum interference and localization in the QKR. Emphasis has been put on the discussion of resonance phenomena occurring

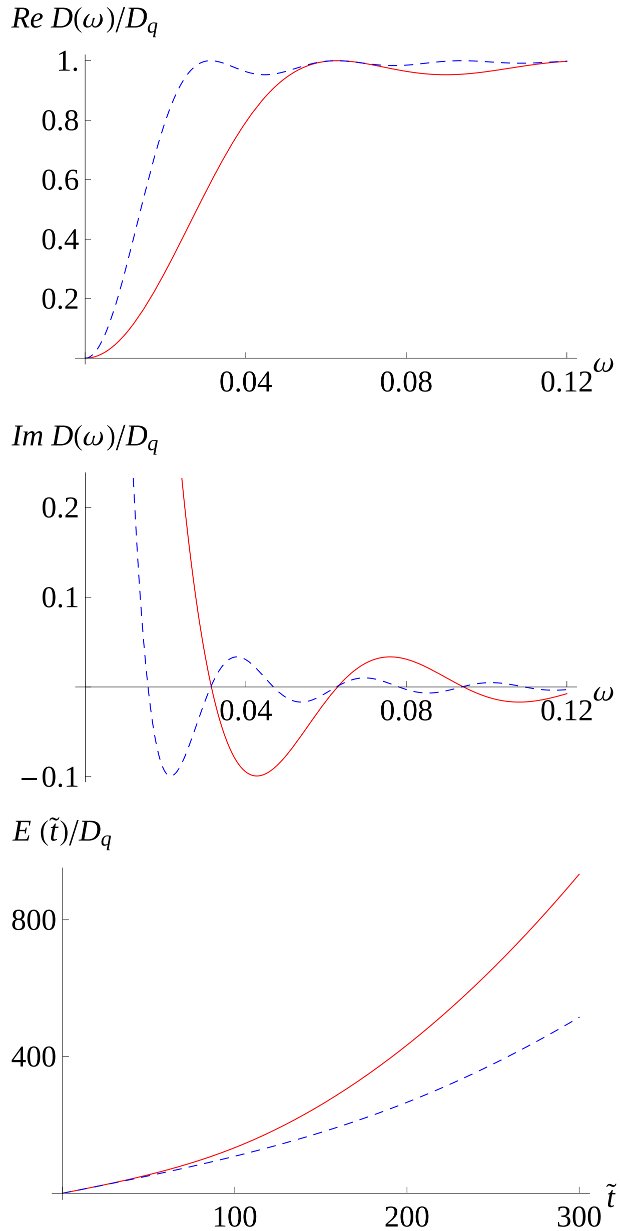

FIG. 2: Real (top) and imaginary (middle) part of the dynamical diffusion constant at resonant values $\tilde{h} = 4\pi p/q$. Bottom: crossover from linear to quadratic energy increase. The parameters are $p = 1, q = 100$ (solid, in red) and $p = 2, q = 201$ (dashed, in blue). In both cases, $K = 5$.

at rational values $\tilde{h} = 4\pi p/q$ of Planck's constant. The prototypical theory [14] did not distinguish between rational and irrational values of $\tilde{h}/4\pi$, a lack of resolution that has prompted criticism [9, 16], and led to the suspicion that the theory was based on hidden ad hoc assumptions. The present generalization is formulated for rational configurations from the outset (the irrational limit can be realized in terms of large coprime values of q and p). It is shown that the system bears similarity to an effectively disordered ring of circumference q and subject to an Aharonov-Bohm flux. (The integration over all values of the flux restores time reversal invariance.) We have obtained quantitative results for the rotor's kinetic energy $E(\tilde{t})$, an observable that carries detailed information on its time-dependent transport characteristics. For periodicities q smaller than the localization length ξ (but larger than the "transport mean free path" $K/\tilde{h}$), we have applied non–perturbative methods to express the full time dependent profile $E(\tilde{t})$ in terms of the universal scaling form Eq. (5). (Earlier results on $E(\tilde{t})$ [6] were formulated in terms of expansions of the long time asymptotics with largely unknown coefficients.) For large values $q > \xi$, we have obtained results for the scaling of

$E(\tilde{t})$ that are quantitative up to corrections in $\exp(-q/\xi)$. The resulting profile conforms with general expectations on energy diffusion in a strongly localized system.

In general, the theory above respects the relevant symmetries of the rotor (time reversal invariance, q-translational invariance, and the duality to near classical dynamics forming close to fundamental resonances at $\tilde{h} = 4\pi p$). Corrections to the diffusion constant are resolved in agreement with earlier work [20, 29, 30]. At the lowest energy scales, the theory resolves the parametric dependence of individual levels in quantitative agreement with exact quantum mechanical calculations. In general, we believe that the criticism of the field theory approach to the rotor [9, 16] has been ill founded: the theory resolves microscopic structures at a resolution no "ad hoc" approach would achieve.

It may be interesting to generalize the approach to other driven quantum systems, notably the two-sided kicked rotor [37], the kicked Haper model [38], the double-kicked particle [39], and the kicked top [40].

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Appendix A: Derivation of Eq. (28)

For $[W, \sigma^3_{\mathrm{AR}}] = 0$ we have $\mathrm{str}\,(W^m) = 0$ for odd m. Therefore, $2\,\mathrm{str}\ln(1 \pm W) = \mathrm{str}\ln(1 - W^2)$. Applying this identity to $S[Z]|_{\omega=0}$ (cf. Eq. (22)) we obtain

$$S[Z]|_{\omega=0} = -\frac{1}{2}\mathrm{str}\left\{ \ln\left[1 - \begin{pmatrix} 0 & Z \\ \tilde{Z} & 0 \end{pmatrix}\right] - \ln\left[1 - \begin{pmatrix} \hat{U} & 0 \\ 0 & \hat{U}^\dagger \end{pmatrix}\begin{pmatrix} 0 & Z \\ \tilde{Z} & 0 \end{pmatrix}\right]\right\}. \tag{A1}$$

Noticing

$$\frac{1}{2}\left(Q\sigma^3_{\mathrm{AR}} + 1\right) = \left[1 - \begin{pmatrix} 0 & Z \\ \tilde{Z} & 0 \end{pmatrix}\right]^{-1},$$

we transform Eq. (A1) to

$$S[Z]|_{\omega=0} = \frac{1}{2}\mathrm{str}\ln\left\{\begin{pmatrix} \frac{\hat{U}^\dagger - 1}{\hat{U}^\dagger + 1} & 0 \\ 0 & -\frac{\hat{U}-1}{\hat{U}+1} \end{pmatrix} Q + 1\right\}.$$

Upon insertion of $\hat{U} = \exp(-i\hat{G})$ this gets us to Eq. (28).

Appendix B: Derivation of Eq. (30)

We first note that the prototype correlation function (15) can be represented as

$$K_\omega(n_1, n_2) = \partial^2_{\alpha\bar{\alpha}}\big|_{\alpha=0} \int D(\bar{\psi}, \psi)\, e^{-\bar{\psi}(G^{-1} + X)\psi},$$

where the source matrix X is defined by

$$X = E^{\uparrow\uparrow} \otimes E^{11}_{\mathrm{BF}} \otimes (\alpha\, E^{12}_{\mathrm{AR}} \otimes \mathcal{P}^{n_2} + \bar{\alpha}\, E^{21}_{\mathrm{AR}} \otimes \mathcal{P}^{n_1}),$$

where, $\mathcal{P}^n_{n_1,n_2} \equiv \delta_{n_1,n}\delta_{n_2,n}$ is a projector onto angular momentum site n and $E^{\uparrow\uparrow}$ projects on the $\uparrow\uparrow$ sector of the theory. Tracing the fate of this source contribution, we see that the action of the functional (19) picks up a contribution

$$\bar{\Psi}_\uparrow(X \otimes E^{11}_{\mathrm{T}} + X^T \otimes E^{22}_{\mathrm{T}})\Psi_\uparrow.$$

Further, the first of the logarithms in (20) gets modified according to

$$\begin{aligned}&\ln(1-\tilde{Z}\tau^{-1}\tilde{Z}^T\tau)\to\\&\to\ln\big(1-(\tilde{Z}+E^{11}_{\rm BF}\otimes(\alpha E^{11}_{\rm T}\otimes\mathcal{P}^{n_2}+\bar{\alpha}E^{22}_{\rm T}\otimes\mathcal{P}^{n_1}))\\&\qquad\times(Z+E^{11}_{\rm BF}\otimes(\bar{\alpha}E^{11}_{\rm T}\otimes\mathcal{P}^{n_1}+\alpha E^{22}_{\rm T}\otimes\mathcal{P}^{n_2}))\big),\end{aligned}$$

where the identification $Z=\tau^{-1}\tilde{Z}^T\tau$ has been used. At this point, we may carry out the derivatives with respect to α and $\bar{\alpha}$ to obtain

$$K_\omega(n_1,n_2)=\big\langle{\rm str}((1-\tilde{Z}Z)^{-1}\tilde{Z}(E^{11}_{\rm BF}\otimes E^{11}_{\rm T}\otimes\mathcal{P}^{n_2}){\rm str}((1-Z\tilde{Z})^{-1}Z(E^{11}_{\rm BF}\otimes E^{11}_{\rm T}\otimes\mathcal{P}^{n_1})\big\rangle,$$

where the angular brackets stand for functional averaging over the functional (22), and we have been using the symmetry (21). In the invariant language of Q–matrices, this assumes the form

$$K_\omega(n_1,n_2)=-\frac{1}{4}\big\langle{\rm str}(Q\,E^{21}_{\rm AR}\otimes E^{11}_{\rm BF}\otimes E^{11}_{\rm T}\otimes\mathcal{P}^{n_2}){\rm str}(Q\,E^{12}_{\rm AR}\otimes E^{11}_{\rm BF}\otimes E^{11}_{\rm T}\otimes\mathcal{P}^{n_1})\big\rangle.$$

Finally, we may resolve the projector matrices $\mathcal{P}^n$ in the Bloch basis. Using Eq. (23), and noting that the Bloch–momentum is 'conserved', i.e. that the phases θ–entering the correlation function are pairwise equal, we thus obtain Eq. (30).

Appendix C: Massive mode fluctuations

In Section IV, we have considered the action of "soft" fluctuations diagonal and long-ranged in angular momentum space. We will here extend the analysis of the action to more general fluctuations. The purpose of this discussion is twofold: First, we will show that massive mode fluctuations feed back into the action of soft fluctuations in that they change their diffusivity. Second, we need to explore the role of massive fluctuations by way of an a posteriori justification of our identification of soft modes above.

As we are going to show below, massive fluctuations are limited in angular momentum space to scales $\Delta n\sim\ell\equiv\frac{K}{\hbar}(\gg 1)$. This is smaller than the resonance periodicity volumes $q>\frac{K}{\hbar}$ considered in this paper. To keep the notation simple, we will therefore neglect the Bloch structure of the fields Z throughout this appendix. In practice, this means that we will work with the basic representation $Z=\{Z_{\alpha tn,\alpha't'n'}\}$ defined in (19).

1. Massive modes

We begin by introducing a few algebraic structures that will facilitate the analysis below. The fields $\tilde{Z}=\{\tilde{Z}_{nn'}\}$ are matrices in the Hilbert space $\mathcal{H}$ of angular momentum states $|n\rangle$. As such, they may be viewed as elements of the Hilbert space $\mathcal{M}\equiv\mathcal{H}\otimes\mathcal{H}^*$, where $\mathcal{H}^*$ is the dual angular momentum space, i.e. the space of states $\{\langle n|\}$. The space $\mathcal{M}$ is spanned by the states,

$$|n,n')\equiv|n\rangle\otimes\langle n'|.$$

We will also need the dual of $\mathcal{M}$, i.e. the space $\mathcal{M}^*$ of states

$$(n,n'|\equiv\langle n|\otimes|n'\rangle.$$

The natural scalar product and resolution of unity in $\mathcal{M}$ are given by, respectively

$$\begin{aligned}(n_1,n_1'|n_2,n_2')&=\delta_{n_1,n_2}\,\delta_{n_1',n_2'},\\\sum_{nn'}|n,n')(n,n'|&=\mathbb{1}_{\mathcal{M}}\,.\end{aligned}\tag{C1}$$

Our analysis of fluctuations below will be conveniently performed in the Fourier transformed basis,

$$|\phi,\sigma) \equiv \sum_n e^{-i\phi n}|n,n-\sigma),$$
$$(\phi,\sigma| \equiv \sum_n \; e^{i\phi n}(n,n-\sigma|,$$

$$|n,n') = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\; e^{i\phi n}|\phi,n-n'),$$
$$(n,n'| = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\,e^{-i\phi n}(\phi,n-n'|. \tag{C2}$$

Eq. (C1) implies the orthogonality and completeness relations

$$(\phi,\sigma|\phi',\sigma') = 2\pi\delta(\phi-\phi')\,\delta_{\sigma,\sigma'},$$
$$\sum_\sigma\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\,|\phi,\sigma)(\phi,\sigma| = \mathbb{1}_{\mathcal{M}}\,. \tag{C3}$$

As discussed in the main text, modes of lowest action have zero angular momentum difference, $\sigma=0$, and wave numbers $\phi<\phi_0$, where ϕ_0 is an upper cutoff to be self-consistently defined. To make the separation of "massive" and "massless" modes explicit, we introduce a projection operator $\hat\pi_0$

$$\hat\pi_0 \equiv \int_s \frac{d\phi}{2\pi}\,|\phi,0)(\phi,0|, \tag{C4}$$

where $\int_s d\phi \equiv \int_{|\phi|<\phi_0} d\phi$. It is straightforward to verify that this is a projection, i.e. $\hat\pi_0^2=\hat\pi_0$. The projection on the complementary sector of "fast modes" is given by $\hat\pi = \mathbb{1}_{\mathcal{M}} - \hat\pi_0$, with the explicit representation

$$\hat\pi = \int_f \frac{d\phi}{2\pi}\,|\phi,0)(\phi,0| + \sum_{\sigma\neq 0}\int\frac{d\phi}{2\pi}\,|\phi,\sigma)(\phi,\sigma|\,, \tag{C5}$$

where $\int_f d\phi \equiv \int_{\phi_0\le|\phi|\le\pi} d\phi$, and $\int d\phi \equiv \int_{-\pi}^{\pi} d\phi$. The projection operation defines a decomposition

$$Z = \hat\pi_0 Z + \hat\pi Z \equiv B + C, \tag{C6}$$

where $B\equiv\hat\pi_0 Z$ and $C\equiv\hat\pi Z$ are the massless and massive contribution to Z, respectively.

2. Renormalized Gaussian action

With these structures in store, we will now re–iterate the derivation of the Gaussian action. However, our discussion will extend that of Section IV A in that we will keep explicit track of C-fluctuations. Ignoring contributions $\mathcal{O}(\omega\partial_n^2)$ (which for the frequencies $\omega \ll 1$ of physical interest are of no relevance), the quadratic action is given by $S[Z] \approx S_{\rm fl}[Z] + S_\omega[Z]$ with

$$S_{\rm fl}[Z] = \frac{1}{2}{\rm str}(\tilde Z Z - \tilde Z\hat U^\dagger Z\hat U)\,, \tag{C7}$$
$$S_\omega[Z] = -\frac{i\omega}{2}\,{\rm str}(\tilde Z Z)\,. \tag{C8}$$

The action S_ω — We first notice that the projector property of $\hat\pi$ implies a decomposition

$$S_\omega[Z] = S_\omega[B] + S_\omega[C].$$

(The proof of this relation is based on the orthogonality property

$${\rm str}(Z\hat\pi\tilde Z) = {\rm str}(Z\hat\pi^2\tilde Z) = {\rm str}(\hat\pi Z\hat\pi\tilde Z), \tag{C9}$$

which in turn follows from the definition (C4).) In principle, the fast field contribution $S_\omega[C]$ leads to a modification of the fast fluctuation action to be discussed below. For the small values of $\omega \lesssim D_0^{-1}$ relevant to our present discussion, however, this modification is irrelevant. We therefore dispose of the contribution $S_\omega[C]$ and turn to the slow action $S_\omega[B]$. Expanding the fields B as

$$B = \int_s \frac{d\phi}{2\pi} B(\phi)|\phi, 0), \tag{C10}$$

where $B(\phi) \equiv (\phi, 0|B = (\phi, 0|Z$ and using the auxiliary identity (a direct consequence of Eq. (C1) and the definition (C2)),

$$\mathrm{tr}(|\phi,\sigma)Z) = (-\phi, -\sigma|Z)e^{-i\phi\sigma},$$

we obtain

$$S_\omega[B] \simeq -\frac{i\omega}{2} \int_s \frac{d\phi}{2\pi} \,\mathrm{str}(B(\phi)\tilde{B}(-\phi)).$$

At this point, we may transform back to angular momentum space. Abbreviating

$$B(n) \equiv B_{n,n},$$

and approximating momentum sums by integrations, $\sum_n \to \int dn$, this gives

$$S_\omega[B] \simeq -\frac{i\omega}{2} \int dn \,\mathrm{str}(B(n)\tilde{B}(n)). \tag{C11}$$

The action $S_{\rm fl}$ — We next turn to the more involved discussion of the fluctuation action. Introducing the notation $\mathrm{Ad}_{\hat{O}} X \equiv \hat{O}^\dagger X \hat{O}$, the latter may be written as

$$\begin{aligned} S_{\rm fl}[B,C] &= \frac{1}{2}\mathrm{str}\left[(\tilde{B}+\tilde{C})\left(1-\mathrm{Ad}_{\hat{U}}\right)(B+C)\right] \\ &= \frac{1}{2}\mathrm{str}\big[\tilde{B}\left(1-\mathrm{Ad}_{\hat{U}}\right)B - \tilde{C}\hat{\pi}\mathrm{Ad}_{\hat{U}}B - \hat{\pi}\mathrm{Ad}_{\hat{U}^{-1}}\tilde{B}C + \tilde{C}\left(1-\hat{\pi}\mathrm{Ad}_{\hat{U}}\right)C\big]\,, \end{aligned} \tag{C12}$$

where the second line is based on (C9).

We next integrate out the massive modes to obtain an effective action $S_{\rm fl}[B]$,

$$e^{-S_{\rm fl}[B]} := \int DC\, e^{-S_{\rm fl}[B,C]}\,. \tag{C13}$$

The Gaussian integration over C can be performed exactly. To fulfill this task, it is convenient to use the saddle point method. The solution of the saddle point equation

$$\frac{\delta S_{\rm fl}[B,C]}{\delta C} = -\hat{\pi}\mathrm{Ad}_{U^{-1}}\tilde{B} + (1-\hat{\pi}\mathrm{Ad}_{U^{-1}})\,\tilde{C} = 0\,,$$

is given by

$$\tilde{C}_{\rm cl} = \left(1-\hat{\pi}\mathrm{Ad}_{\hat{U}^{-1}}\right)^{-1}\hat{\pi}\mathrm{Ad}_{\hat{U}^{-1}}\tilde{B}.$$

Applying Eq. (21) to this equation, we obtain

$$C_{\rm cl} = \left(1-\hat{\pi}\mathrm{Ad}_{\hat{U}}\right)^{-1}\hat{\pi}\mathrm{Ad}_{\hat{U}}B.$$

As usual in supersymmetric theories the integration over quadratic fluctuations around the saddle point gives a factor of unity, i.e. the effective action is obtained by straightforward substitution of the saddle point configurations $(Z^m_{\rm cl}, \tilde{Z}^m_{\rm cl})$ into the action (C12). As a result, we obtain

$$S_{\rm fl}[B] = \frac{1}{2}\mathrm{str}\left(\tilde{B}\left(1-\mathrm{Ad}_{\hat{U}}\frac{1}{1-\hat{\pi}\mathrm{Ad}_{\hat{U}}}\right)B\right),$$

In the following, it will be convenient to split the action into two contributions,

$$\begin{aligned} S_{\rm fl}[B] &= S^0_{\rm fl}[B] + S^m_{\rm fl}[B], \\ S^0_{\rm fl}[B] &= \frac{1}{2}\mathrm{str}\left(\tilde B\left(1-\mathrm{Ad}_{\hat U}\right)B\right), \\ S^m_{\rm fl}[B] &= -\frac{1}{2}\mathrm{str}\left(\tilde B\left(\mathrm{Ad}_{\hat U}\frac{\hat\pi \mathrm{Ad}_{\hat U}}{1-\hat\pi\mathrm{Ad}_{\hat U}}\right)B\right), \end{aligned} \tag{C14}$$

where the two terms represent the native slow mode action ($S^0_{\rm fl}$) and its renormalization by the inclusion of massive modes ($S^m_{\rm fl}$), respectively. The structure of the action (C14) solely relies on the projector property of $\hat\pi$, but not on details of the projection operation. We next aim to bring the action into a more explicit form, and to this end, we need to take the dynamical structure of the Floquet operator into account.

Before turning to the calculation, it may be worthwhile to discuss the physical meaning of the action $S^m_{\rm fl}$. This action describes the renormalization of the soft mode action by "massive" fluctuations. Consider a generic massive fluctuation mode $C_{n_1n_2}$, non-diagonal in angular momentum space. As discussed in Section IV A, we may loosely interpret $C_{n_1n_2}$ as the representative of quantum states $|n_1\rangle\langle n_2|$. Passing to the Wigner representation, the off-diagonality, i.e., $\Delta n = n_1 - n_2 \neq 0$, leads to an excitation in the angular direction of the "phase space". As we will see, the kernel $(1-\hat\pi\mathrm{Ad}_{\hat U})^{-1}$ describes the ways in which the memory of initial "phase space" information is maintained. For generic irrational values $\tilde h \ll K$, these effects diminish rapidly. (One may construct certain infinite fraction representations of $\tilde h$ which sum up to irrational values and, nonetheless, lead to long-ranged angular correlations [23]. At this point, we are not able to get the ensuing memory kernels under control.) The inclusion of this memory kernel will lead to a renormalization of the diffusion constant of the theory. Equally important, it may affect the length scales at which the dynamics becomes genuinely diffusive. Again, the situation bears analogy to the physics of disordered metals. There, wave packets initially centered around a definite momentum maintain the memory of their direction of propagation over a certain time scale, the so-called transport scattering time. In a manner largely analogous to our present discussion, the transport scattering time can be seen to determine an effective diffusion constant.

The action $S^0_{\rm fl}[B]$ — We first consider the unrenormalized massless action S^0. The derivation of this action essentially repeats the construction of Section IV A in the present notation and introduces technical building blocks relevant to the discussion below. Specifically, we will need the matrix elements of the adjoint Floquet operator in the two bases $\{|n,n')\}$ and $\{|\phi,\sigma)\}$. Using that $(n_+,n_-|\mathrm{Ad}_{\hat U}|n'_+,n'_-) = \langle n_+|\hat U|n'_+\rangle\langle n'_-|\hat U^\dagger|n_-\rangle$, a straightforward if somewhat tedious calculation obtains

$$\begin{aligned} (n_+,n_-|\mathrm{Ad}_{\hat U}|n'_+,n'_-) &= i^{n_+-n'_+-n_-+n'_-}e^{i\frac{\tilde h}{2}(n_+^2-n_-^2)}J_{n_+-n'_+}\left(\frac{K}{\tilde h}\right)J_{n'_--n_-}\left(-\frac{K}{\tilde h}\right), \\ (\phi,\sigma|\mathrm{Ad}_{\hat U}|\phi',\sigma') &= 2\pi\tilde\delta(\phi-\phi'+\tilde h\sigma)e^{-i\frac{\tilde h}{2}\sigma^2+i\frac{\phi'}{2}(\sigma-\sigma')}J_{\sigma-\sigma'}\left(\frac{2K}{\tilde h}\sin\left(\frac{\phi'}{2}\right)\right). \end{aligned} \tag{C15}$$

where $\tilde\delta(x)$ stands for that the δ–function holds only modulo 2π, and

$$J_n(z) = \frac{1}{i^n}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{iz\cos\theta}e^{in\theta}. \tag{C16}$$

are the Bessel functions of the first kind. Using this result and representing the slow fields as (C10), we then obtain

$$S^0_{\rm fl}[B] = \frac{1}{2}\int\limits_s\frac{d\phi}{2\pi}\frac{d\phi'}{2\pi}\mathrm{str}(B(\phi)\tilde B(\phi'))(-\phi,0|1-\mathrm{Ad}_{\hat U}|\phi',0).$$

Using Eqs. (C3) and (C15) this becomes

$$S^0_{\rm fl}[B] = \frac{1}{2}\int_s\frac{d\phi}{2\pi}\mathrm{str}(B(\phi)\tilde B(-\phi))\left(1-J_0\left(\frac{2K}{\tilde h}\sin\left(-\frac{\phi}{2}\right)\right)\right). \tag{C17}$$

This action will describe diffusive correlations of the field B, provided the angular kernel $(1-J_0)$ can be approximated as a *quadratic* function of ϕ. This is the case, if ϕ is sufficiently small. Specifically, for a soft mode angular cutoff

$$\phi_0 \ll \ell^{-1} = \frac{\tilde h}{K}, \tag{C18}$$

with $\tilde{h}/K$ sufficiently small, we have $\frac{2K}{\tilde{h}}|\sin\frac{\phi}{2}| \ll 1$ which means that the Bessel function may be expanded as

$$J_0(x) \stackrel{x\ll 1}{=} 1 - \frac{x^2}{4} + \mathcal{O}(x^4),$$

to arrive at $S^0_{\text{fl}}[Z] \simeq \frac{K^2}{8\tilde{h}^2}\int \frac{d\phi}{2\pi}\phi^2 \text{str}(B(\phi)\tilde{B}(-\phi))$. Finally, transforming back to angular momentum space, we obtain

$$S^0_{\text{fl}}[B] \simeq \frac{D_0}{2}\int dn\, \text{str}(\partial_n B \partial_n \tilde{B}). \tag{C19}$$

Notice that the condition (C18) stabilizing this two-derivative action has an analog in the physics of disordered metals. There, the wave numbers of diffusive fluctuations (fluctuations governed by a second order derivative evolution operator) are smaller than the inverse mean free path, l_{tr}, due to impurity scattering. In the present context, the scale $K/\tilde{h}$ plays a role analogous to l_{tr} and modes fluctuating at larger length scales evolve diffusively.

The action $S^m_{\text{fl}}[B]$ — We next discuss the second contribution S^m_{fl} to the fluctuation action (C14). Inserting the expansion $B = \int_s \frac{d\phi}{2\pi} B(\phi)|\phi, 0)$ into Eq. (C14) and expanding the resolvent operator $(1 - \hat{\pi}\text{Ad}_{\hat{U}})^{-1}$ we obtain our starting point

$$S^m_{\text{fl}} = -\frac{1}{2}\int_s \frac{d\phi}{2\pi}\frac{d\phi'}{2\pi}\text{str}(B(-\phi)\tilde{B}(\phi'))\sum_{n=1}^{\infty}(\phi, 0|\text{Ad}_{\hat{U}}(\hat{\pi}\text{Ad}_{\hat{U}})^n|\phi', 0).$$

We now insert resolutions of unity (C3) between the n factors $\hat{\pi}\text{Ad}_{\hat{U}}$, use that the projectors $\hat{\pi}$ appearing are diagonal in the basis $\{|\phi, \sigma)\}$, i.e. $\hat{\pi}|\phi, \sigma) \equiv \pi(\phi, \sigma)\,|\phi, \sigma)$, where $\pi(\phi, \sigma) = \delta_{\sigma,0}\theta(|\phi| - \phi_0) + (1 - \delta_{\sigma,0})$, ($\theta(x)$ is the Heaviside function, which is unity for $x \geq 0$ and zero otherwise.) and substitute the explicit form of the matrix elements (C15) to arrive at the "path integral" representation

$$\begin{aligned} S^m_{\text{fl}}[Z] = &-\frac{1}{2}\int_s \frac{d\phi}{2\pi}\frac{d\phi'}{2\pi}\text{str}(B(-\phi)\tilde{B}(\phi')) \\ &\times \sum_{n=1}^{\infty}\int D(\phi, \sigma)\prod_{i=1}^{n+1} 2\pi\tilde{\delta}(\phi_i - \phi_{i-1} + \tilde{h}\sigma_i) e^{i\sum_{i=1}^{n+1}(-\frac{\tilde{h}}{2}\sigma_i^2 + \frac{\phi_i}{2}(\sigma_i - \sigma_{i-1}))} J_{\sigma_i - \sigma_{i-1}}\left(\frac{2K}{\tilde{h}}\sin\left(\frac{\phi_{i-1}}{2}\right)\right), \end{aligned} \tag{C20}$$

Here, the integration measure is given by

$$\int D(\phi, \sigma) \equiv \prod_{i=1}^{n}\int_0^{2\pi} d\phi_i \sum_{\sigma_i}\pi(\phi_i, \sigma_i),$$

and the boundary conditions $(\phi_{n+1}, \sigma_{n+1}) = (\phi, 0)$ and $(\phi_0, \sigma_0) = (\phi', 0)$ are understood. We may now use the $\tilde{\delta}$-function constraint in the integral to collapse the phase factor to multiple 2π,

$$\sum_{i=1}^{n+1}\left(\tilde{h}\sigma_i^2 - \phi_i(\sigma_i - \sigma_{i-1})\right) \equiv 0 \bmod 2\pi.$$

The constraint further implies

$$\phi_i \equiv \phi' - \tilde{h}\sum_{j=1}^{i}\sigma_j \bmod 2\pi. \tag{C21}$$

For generic irrational $\tilde{h}/(4\pi)$ (for "non–generic" values, see Ref. [23]), ϕ_i is not close to a multiple 2π which implies $\frac{2K}{\tilde{h}}\sin(\phi_i/2) = \mathcal{O}(\frac{K}{\tilde{h}}) \gg 1$. Now for large arguments,

$$J_\sigma(x) \stackrel{|x|\gg 1}{\sim} \frac{1}{\sqrt{x}},$$

which means that Eq. (C20) is a controlled expansion in $(\tilde{h}/K)^{1/2} \ll 1$. We may truncate this expansion at any desired order n. In conjunction with the near irrationality of $\tilde{h}/4\pi$, the finiteness of the sum $\sum_{j=1}^{n+1}\sigma_j \in \mathbb{N}$ means that $\left(\tilde{h}\sum_{j=1}^{n+1}\sigma_j\right) \bmod 2\pi$, can not be of $\mathcal{O}(\phi') < \mathcal{O}(\tilde{h}/K)$ unless $\sum_{j=1}^{n+1}\sigma_j = 0$. We thus conclude

$$\sum_{j=1}^{i}\sigma_j = 0, \tag{C22}$$

giving $\phi = \phi'$: the integral over intermediate states conserves the momentum ϕ.

The considerations above fix the simplified action

$$S_{\text{fl}}^{m}[Z] = -\frac{1}{2}\int_s \frac{d\phi}{2\pi}\,\text{str}(B(-\phi)\tilde{B}(\phi))\sum_{n=1}^{\infty}\sum_{\{\sigma\}}\prod_{i=1}^{n+1} J_{\sigma_i-\sigma_{i-1}}\left(\frac{2K}{\tilde{h}}\sin\left(\frac{\phi_{i-1}}{2}\right)\right), \tag{C23}$$

where ϕ_i is given by (C21) and $\sum_{\{\sigma\}} = \prod_i \sum_{\sigma_i}$, subject to the constraint (C22).

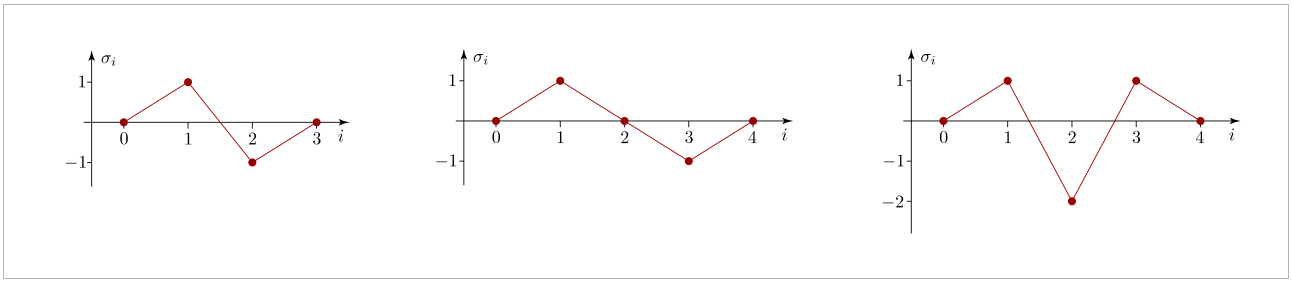

FIG. 3: Path of lowest order contributing to the renormalization of the diffusion constant.

It is instructive, to think of the set $\{(\phi_i, \sigma_i)\}$ in terms of a path

$$(\phi, 0) \to (\phi - \tilde{h}\sigma_1, \sigma_1) \to (\phi - \tilde{h}\sigma_1 - \tilde{h}\sigma_2, \sigma_2) \to \cdots \to (\phi - \tilde{h}\sigma_1 - \cdots - \tilde{h}\sigma_{n-1}, \sigma_{n-1}) \to (\phi, \sigma_n) \to (\phi, 0). \tag{C24}$$

We may imagine the $n+1$ links "$\to$" of this path to be weighted by Bessel functions. Specifically, the first and last link, respectively carry weights $J_{\sigma_1}\left(\frac{2K}{\tilde{h}}\sin\left(\frac{\phi}{2}\right)\right)$ and $J_{-\sigma_n}\left(\frac{2K}{\tilde{h}}\sin\left(\frac{\phi}{2}\right)\right)$. These weights imply a further effective constraint on the sum over configurations $\{\sigma\}$: since $\phi < \phi_0 \ll \tilde{h}/K$, the terminating Bessel functions carry arguments $\frac{2K}{\tilde{h}}|\sin\frac{\phi}{2}| \simeq |K\phi/\tilde{h}| \ll 1$. We may thus apply the asymptotic formula

$$J_\sigma(\epsilon) \overset{|\epsilon|\ll 1}{=} \frac{\epsilon^{|\sigma|}}{2^{|\sigma|}\sigma!} + \ldots,$$

to conclude that the dominant contribution to the sum has

$$|\sigma_1| = |\sigma_n| = 1. \tag{C25}$$

The terminal weights then yield a factor

$$J_{\sigma_1}\left(\frac{2K}{\tilde{h}}\sin\left(\frac{\phi}{2}\right)\right) J_{-\sigma_n}\left(\frac{2K}{\tilde{h}}\sin\left(\frac{\phi}{2}\right)\right) \simeq -\left(\frac{K\phi}{2\tilde{h}}\right)^2 \text{sgn}(\sigma_1\sigma_n).$$

For the arguments of the inner Bessel functions, regardless of the value of $\tilde{h}$, they much exceed unity, which means that the sum above will be dominated by the shortest paths obeying the criteria (C22) and (C25).

The contributions of leading and sub–leading order to the path sum are shown in Fig. 3. Evaluating these paths in (C23), we obtain

$$S_{\text{fl}}^{m}[B] = \frac{D_0}{2}\int_s \frac{d\phi}{2\pi}\,\phi^2\,\text{str}(B(-\phi)\tilde{B}(\phi))\left(-2J_2(K_q) - 2J_1^2(K_q) + 2J_3^2(K_q)\right). \tag{C26}$$

Passing back to angular momentum space and adding to S_{fl}^{m} the compounds (C19) and (C11), we arrive at the final form of the Gaussian action

$$S[B] = \frac{1}{2}\int dn\,\text{str}(B(n)(-D_q\partial_n^2 - i\omega)\tilde{B}(n)),$$

where the diffusion constant is given by (36). Corrections to the above diffusion constant can be computed by including path of higher complexity and will be of $\mathcal{O}(K_q^{-3/2})$. However, at this order in the expansion, anharmonic fluctuations

of $\mathcal{O}(\mathrm{str}(B\tilde{C}B\tilde{C}))$ begin to play a role and the computation becomes significantly more complicated (see Ref. [20] for the form of the ensuing corrections.)

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